General Relativity Exam Problem 1

Is the induced map on tangent spaces linear?

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Problem Statement

Recall that when M and N are smooth manifolds, $\phi : M \to N$ is a diffeomorphism, and p is a point in M, there is an induced map on the tangent spaces $T_p\phi : T_p(M) \to T_{\phi(p)}(N)$ as defined in lecture. Show that $T_p\phi$ is linear.

Solution

Recall that when M and N are smooth manifolds, $\phi : M \to N$ is a diffeomorphism, and p is a point in M, there is an induced map on the tangent spaces $T_p\phi : T_p(M) \to T_{\phi(p)}(N)$ as defined in lecture. Show that $T_p\phi$ is linear.

Solution

Recall how the map $T_p\phi$ is defined:

$$\begin{aligned} T_p\phi : T_p(M) &\longrightarrow T_{\phi(p)}(N) \\ \xi &\longmapsto \xi \circ \phi^* \end{aligned}$$

With $f \in \mathcal{F}(\phi(p))$ we use the induced map ϕ^* as follows.

$$\begin{aligned} \big[T_p\phi(\alpha\xi_1 + \beta\xi_2)\big](f) &= \Big[(\alpha\xi_1 + \beta\xi_2) \circ \phi^*\Big](f) \\ &= (\alpha\xi_1 + \beta\xi_2)\big(f \circ \phi\big) \\ &= \alpha(\xi_1 \circ f \circ \phi) + \beta(\xi_2 \circ f \circ \phi) \\ &= \alpha(\xi_2 \circ \phi^*)(f) + \beta(\xi_2 \circ \phi^*)(f) \\ &= \alpha\big[T_p\phi(\xi_1)\big](f) + \beta\big[T_p(\xi_2)\big](f) \end{aligned}$$