General Relativity Exam Problem 2

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AMATH 875

Problem Statement

Prove $\mathsf{GL}(n\,;\mathbb{R})$ is a smooth manifold, and compute $T_{\mathbb{1}}(\mathsf{GL}(n\,;\mathbb{R}))$ using the geometric definition of the tangent space.

Is $\mathsf{GL}(n\,;\mathbb{R})$ a manifold?

Definition

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- $\mathrm{GL}(n\,;\mathbb{R}) = \det^{-1}(\mathbb{R} \setminus \{0\})$
- Thus $\mathrm{GL}(n\,;\mathbb{R})$ is an open subset of a smooth manifold

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- Thus $T_{\mathbb{1}}(\mathsf{GL}(n\,;\mathbb{R})) = \mathbb{R}^{n\times n}$