General Relativity Exam Problem 3

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AMATH 875

Topology of Minkowski Space

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Is the topology of Minkowski space the same as that of $\mathbb{R}^4$? My thoughts would be no, because of the very different inner products define very different metrics, and because the metric determines the open balls, it determines the topology.

special-relativity mathematical-physics topology

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asked Apr 13 '17 at 15:09

Nate Stemen

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Does the topology on Minkowski space agree with the "metric topology" induced by the metric $\eta_{\mu\nu} = \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$?

What is the topology on $\mathbb{R}^{1,3}$ with the metric $\eta_{\mu\nu} = \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$?

What's Minkowski space?

$\mathbb{R} \times \mathbb{R}^3$ equipped with the metric

$$\eta_{\mu\nu} = \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}.$$

Recall: What's the metric tensor?

Definition

Let M be a smooth manifold. The metric tensor g is a rank $(0, 2)$ tensor field that is

- bilinear
- symmetric
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Another way of viewing g is that for each $p \in M$ we have a nondegenerate symmetric bilinear form $g_p : T_p(M) \times T_p(M) \to \mathbb{R}$ that varies smoothly as we move p.

What happens when $T_p(M) \cong M$ for all p?

- If $T_p(M) \cong M$ then our metric tensor is extended to all of M: $g_p : M \times M \to \mathbb{R}$
- Can drop subscript p
- Hence we have a nondegenerate, symmetric bilinear form on all of M
- Sounds like an inner product?
- Can then define a norm: $\|u\|_g \stackrel{\text{def}}{=} g(u, u)$
- Can then define a metric: $d_{\|\cdot\|_g}(x, y) \stackrel{\text{def}}{=} \|x - y\|_g$

What is the metric topology?

Definition (Metric Space)

A *metric space* is a pair (M, d) where M is a set, and d is a function $d : M \times M \to \mathbb{R}$ satisfying

- $d(x, y) = 0$ if and only if $x = y$,
- $d(x, y) = d(y, x)$, and
- $d(x, z) \leq d(x, y) + d(y, z)$.

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Definition (Metric Topology)

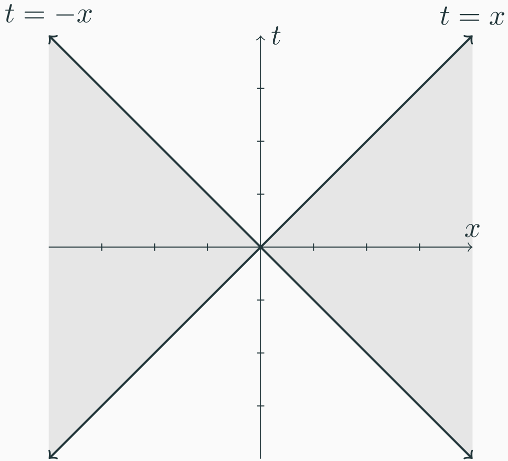
For any $x \in M$ and $r > 0$, we define the open ball of radius r to be

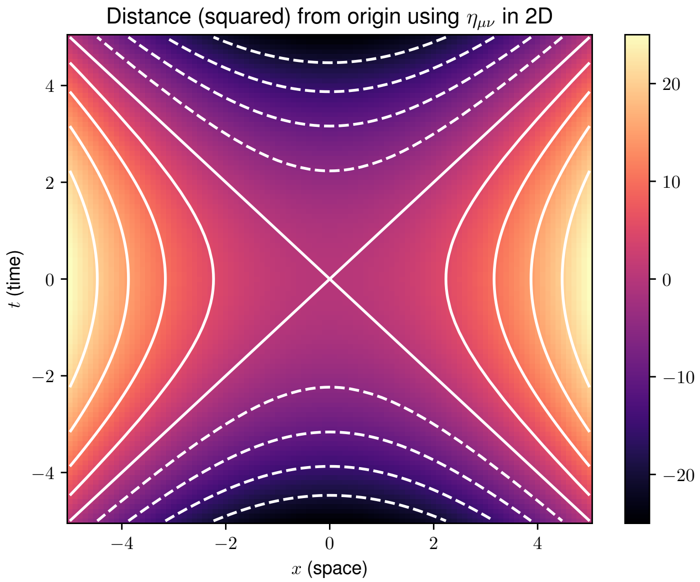
$$B(x\,; r) = \{y \in M : d(x, y) < r\}.$$

These open balls form a base for a topology.

$\mathbb{R}^n$ (the set) taken with the open balls generated by the Euclidean metric form the standard topology for $\mathbb{R}^n$ (the topological space).

Open Balls in Minkowski Space





Distance (squared) from origin using $\eta_{\mu\nu}$ in 2D
t (time)
x (space)
4
2
0
−2
−4
20
10
0
−10
−20

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- I thought the metric might have to coincide with the topology, but perhaps not