General Relativity for Cosmology Lectures 5

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875

In this lecture we saw two important topics:

1. the inner derivation on the algebra of completely antisymmetric differential forms
2. the definition of the Lie derivative

As we saw in the last lecture, the exterior derivative $\mathrm{d} : \Lambda^k(M) \to \Lambda^{k+1}(M)$ gave us a way to differentiate rank $(0, k)$ covariant tensors. That said, in order to do more advanced physics, and presumably general relativity, we need a way to differentiate more advance objects, such as any rank (a, b) tensor with respect to any other. To this end we construct the inner derivation $\iota_X : \Lambda^k(M) \to \Lambda^{k-1}(M)$ and use it in conjunction with the exterior derivative to arrive at a definition of derivative that is general enough to handle our wildest dreams.

Item 1: We first *define* the inner derivation to send 0-forms to 0, and 1-forms to a 0-form under $\omega \mapsto \iota_\xi(\omega) = \omega(\xi)$ where ξ is vector field defined on our manifold M. With this definition we then saw the way this map works if $\gamma \in \Lambda^k(M)$ then we have

$$\iota_\xi(\gamma)(\eta_1, \ldots, \eta_{k-1}) = \gamma(\xi, \eta_1, \ldots, \eta_{k-1}).$$

Some call this map a contraction as we are simply using ξ as the first positional argument.

Item 2: We now have a map that raises the degree of the form (the exterior derivative d), and a map that lowers the degree (the interior/inner derivative ι_ξ). Our goal is to construct a degree-0 derivation on $\Lambda(M)$ such that it extends the notion of derivative, and aligns with our previous notions of directional derivatives on functions (0-forms). This goal is accomplished by taking

$$\mathcal{L}_\xi := \mathrm{d} \circ \iota_\xi + \iota_\xi \circ \mathrm{d}.$$

Well indeed this does what we want, but if we define the Lie derivative this way, then it won't actually be defined on all tensor fields because d and ι_ξ are not defined on all tensor fields. Hence we go back to our roots and the first definition of the derivative we encounter: the Newton-Leibniz definition of a derivative. There are some complications here since a tensor field at a point p and a nearby point $p + \varepsilon$ cannot be subtracted due to living in completely different spaces. The first way I thought this would be solved is via parallel transport, and maybe that's what Achim is doing, but using a slightly different language of "flows" instead.