Lie Groups and Lie Algebras Assignment 1

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1

Let $G \subset \mathrm{GL}(n; \mathbb{C})$ and $H \subset \mathrm{GL}(n; \mathbb{C})$ be matrix Lie groups. Consider the following set of block diagonal matrices.

$$\widetilde{G} := \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in \mathcal{M}_{n+m}(\mathbb{C}) \mid A \in G, B \in H \right\}$$

Prove that this is a matrix Lie group. Then prove that $\widetilde{G} \simeq G \times H$ as groups and topological spaces, where the product topology is put on $G \times H$.

Solution. First we will show this is a matrix Lie group by taking a sequence $\{A_i\}_{i\in\mathbb{N}} \in \widetilde{G}$. The structure of $\widetilde{G}$ allows us to understand each term in this sequence as

$$A_i = \begin{pmatrix} B_i & 0 \\ 0 & C_i \end{pmatrix}.$$

Thus, every sequence $\{A_i\}_{i\in\mathbb{N}} \in \widetilde{G}$ is comprised of two sequences $\{B_i\}_{i\in\mathbb{N}} \in G$ and $\{C_i\}_{i\in\mathbb{N}} \in H$. The fact that G and H are both Lie groups allow us to conclude $\lim B_i = B \in G$ and $\lim C_i = C \in H$, and thus $\lim A_i = \left(\begin{smallmatrix} B & 0 \\ 0 & C \end{smallmatrix}\right) \in \widetilde{G}$. Thus we conclude $\widetilde{G}$ is indeed a Lie group.

To show the two groups are isomorphic, take $\phi : \widetilde{G} \to G \times H$ by

$$\phi(A) = \phi\left(\begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix}\right) \mapsto (B, C).$$

First note this is defined on all of $\widetilde{G}$, and is indeed a bijection by the definition of $\widetilde{G}$. Now we'll show it's a homomorphism.

$$\phi(AB) = \phi\left(\begin{bmatrix} C & 0 \\ 0 & D \end{bmatrix}\begin{bmatrix} E & 0 \\ 0 & F \end{bmatrix}\right) = \phi\left(\begin{bmatrix} CE & 0 \\ 0 & DF \end{bmatrix}\right) = (CE, DF)$$
$$\phi(A)\phi(B) = (C, D) \times (E, F) := (CE, DF)$$

Lastly we must show ϕ to be a homeomorphism. First note $\phi^{-1} : G \times H \to \widetilde{G}$ can be defined as $\phi^{-1}((A, B)) = \left[\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right]$, and the continuity of both ϕ and ϕ^{-1} follow from the continuity of matrix multiplication.

2

Let $\alpha \in \mathbb{R}$ be irrational.

(a) Prove that the set $\{e^{2\pi i n\alpha} \mid n \in \mathbb{Z}\}$ is dense in S^1.

(b) Define

$$G = \left\{ \begin{pmatrix} e^{it} & 0 \\ 0 & e^{i\alpha t} \end{pmatrix} \mid t \in \mathbb{R} \right\}.$$

Prove that $\overline{G}$, the closure of G in $\mathcal{M}_2(\mathbb{C})$, satisfies

$$\overline{G} = \left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{i\phi} \end{pmatrix} \mid \theta, \phi \in \mathbb{R} \right\}.$$

(c) Is G a matrix Lie group? What about $\overline{G}$.

Solution. (a) First we will show $A := \{e^{2\pi i n\alpha} \mid n \in \mathbb{Z}\}$ is a group under complex number multiplication.

$$e^{2\pi i n\alpha} e^{2\pi i m\alpha} = e^{2\pi i(n+m)\alpha} \in A$$

The identity is given by taking $n = 0$, and inverses are taking by $-n$ yada yada yada... Also note this set/group has cardinality that is countably infinite, because of the irrationality of α.

Now divide S^1 into N equally sized bins, as if slicing a pizza. By the pidgeonhole principle, one such slice must contain an infinite number of points. In particular we can find two elements x, y in that such slice so that $\left|x \cdot y^{-1}\right| < \varepsilon_N$. We can then use this element $x \cdot y^{-1}$ to generate an ε-net of the unit circle. Because this ε is dependent on N, we can shrink it as small as we want, and hence generate points within any ε of S^1. Thus this set is dense in S^1.

(b) Let's construct two sequences. First take

$$g_n = \begin{pmatrix} e^{i(\theta+2\pi n)} & \\ & e^{i\theta\alpha} e^{i2\pi\alpha n} \end{pmatrix} = \begin{pmatrix} e^{i\theta} & \\ & e^{i\theta\alpha} e^{i2\pi\alpha n} \end{pmatrix}.$$

Now that the *subsequence* of g_n so that the second term converges to 1. This can always be done by Part (a). Similarly we will take

$$h_n = \begin{pmatrix} e^{i\left(\frac{\phi+2\pi n}{\alpha}\right)} & \\ & e^{i\alpha\left(\frac{\phi+2\pi n}{\alpha}\right)} \end{pmatrix} = \begin{pmatrix} e^{i\phi/\alpha} e^{i\beta 2\pi n} & \\ & e^{i\phi} \end{pmatrix}.$$

We've used $\beta = 1/\alpha$ (which if α is irrational will still be irrational). Again by Part (a) we can find a subsequence of of h_n to converge to $\begin{pmatrix} 1 & \\ & e^{i\phi} \end{pmatrix}$.

By multiplying these two sequences together we get all elements like $\begin{pmatrix} e^{i\theta} & \\ & e^{i\phi} \end{pmatrix}$.

(c) G is not a Lie group because it is not relatively closed in $\mathcal{M}_2(\mathbb{C})$. That said $\overline{G}$ because first it is a subgroup of $\mathcal{M}_2(\mathbb{C})$ and it *is* relatively closed.

3

Let G be a matrix Lie group. The following problems are not necessarily related.

(a) Suppose G has a dense abelian subgroup, prove that G itself is abelian.

(b) Assume G is connected and let H be a discrete normal subgroup of G. Prove that H is contained in the center $Z(G)$ of G.

(c) Assume G is connected and let U be a neighborhood of the identity $\mathbb{1}$. Prove that every element $A \in G$ can be written as $A = A_1 A_2 \cdots A_n$ for some $n \in \mathbb{N}$ and $A_1, \ldots, A_n \in U$.

Solution. (a) Call the dense subgroup H. Take two element $g, h \in G$ which are not in H. The density of H lets us write g and h as limits of elements in H.

$$\begin{aligned} g \cdot h &= \lim_{i\to\infty} g_i \cdot \lim_{j\to\infty} h_j \\ &= \lim_{i,j\to\infty} g_i \cdot h_j \\ &= \lim_{i,j\to\infty} h_j \cdot g_i \\ &= \lim_{j\to\infty} h_j \cdot \lim_{i\to\infty} g_i \\ &= h \cdot g \end{aligned}$$

A similar analysis can be done if one element is not in H, and another is.

(b) Since H is normal we know $ghg^{-1} \in H$ for al $g \in G$. In particular this must hold for $g = e_G$ the identity in G, and hence $h \in H$. By the discreteness of H we know there is a neighborhood U around h such that $U \cap H = \{h\}$. This fact, combined with the continuity of multiplication in Lie groups allows us to say $ghg^{-1} \in U \cap H = \{h\}$. Thus $ghg^{-1} = h$ and by right multiplying by g we have $gh = hg$. This implies H is contained in the center $Z(G)$ of G.

(c) Here we will make use of the fact that any open and closed subgroup H of a connected Lie group G must be equal $H = G$.

First take U to be the intersection $U \cap U^{-1}$. This is still an open neighborhood of the identity because the inversion map $g \mapsto g^{-1}$ is smooth. Now build the group

$$H = \bigcup_{n\in\mathbb{N}} U^n = \{u_1 \cdot u_2 \cdots u_n : u_i \in U \text{ and for some } n \in \mathbb{N}\}.$$

Since each U^n is open, H must also be open because it is the union of open sets.

To show this set is closed, take and element $b \in \overline{H}$ the closure of H. Since bU^{-1} is open, it must intersect H and thus we can find an $h \in H \cap gU^{-1}$. This means $h = gu^{-1}$ for some $u \in U$, and $h = u_1 \cdot u_2 \cdots u_n$ for some $u_i \in U$. Setting these two representation equal we can say $g = u_1 \cdot u_2 \cdots u_n \cdot u \in U^{n+1} \subseteq H$. Thus H is closed. Finally using the statement from the beginning of this problem of the solution we conclude that G is generated by U.

4

Prove that $\mathsf{SO}(n)$ is connected for all $n \geq 1$.

Solution. First note that $\mathsf{SO}(1) = \{[1]\}$ is connected. Revolutionary.

Now for any unit vector $v \in \mathbb{R}^n$, take $\mathbf{e}_1$ to be the first standard basis vector and pick $\mathbf{e}_2$ to be orthogonal to $\mathbf{e}_1$ and with the property that $v \in \mathrm{span}(\mathbf{e}_1, \mathbf{e}_2)$. Complete the basis arbitrarily. The angle between $\mathbf{e}_1$ and v can be computed, and we call it ϕ. Our path can be constructed as:

$$p(t) = \begin{bmatrix} \cos\phi t & \sin\phi t & \\ -\sin\phi t & \cos\phi t & \\ & & \mathbb{1}_{n-2} \end{bmatrix}$$

This is clearly in $\mathsf{SO}(n)$ and is a path that rotates $\mathbf{e}_1$ to v.

Since the rotation part of the above matrix is in $\mathsf{SO}(2)$, we can do an orientation preserving change of basis (which will also be in $\mathsf{SO}(2)$) to transform the above path into

$$\begin{bmatrix} 1 & & \\ & R_{\phi t} & \\ & & \mathbb{1}_{n-3} \end{bmatrix} = \begin{bmatrix} 1 & \\ & A \end{bmatrix}.$$

Where $A \in \mathsf{SO}(n-1)$. By induction this shows that every element in $\mathsf{SO}(n)$ can be connected to the identity, and hence is connected.

5

An alternative proof of the connectedness of $\mathsf{GL}(n;\mathbb{C})$.

(a) Let $A, B \in \mathsf{GL}(n;\mathbb{C})$. Prove that there are only finitely many $\lambda \in \mathbb{C}$ such that $\det(\lambda A + (1-\lambda)B) = 0$.

(b) Prove that there is a continuous function $\lambda : [0,1] \to \mathbb{C}$ with $\lambda(0) = 0, \lambda(1) = 1$, such that $A(t) = \lambda(t)A + (1-\lambda(t))B$ lies in $\mathsf{GL}(n;\mathbb{C})$ for all $t \in [0,1]$. Deduce that $\mathsf{GL}(n;\mathbb{C})$ is connected.

(c) Were does this argument fail when $\mathbb{C}$ is replaced with $\mathbb{R}$.

Solution. (a) First note that the determinant is a continuous function because it can be written as a polynomial in the entries of a matrix. This means there exists a neighborhood around B (and A) such that the determinat of $\varepsilon A + (1-\varepsilon)B$ must be approximately the same as the determinant of B (and in particular, nonzero). This fact, together with the determinant being a polynomial allow us to conclude there are only finite number of roots of this function on the line joining A and B.

(b) There always exists a continuous paths connecting any two matrices in $\mathsf{GL}(n;\mathbb{C})$ because there are an uncountable number of paths, and only a finite number of points to avoid. Clearly this can be done.

(c) This argument fails when dealing with $\mathsf{GL}(n;\mathbb{R})$ because we cannot "go around" the holes because the determinant maps to $\mathbb{R}$ (a one dimensional space with a hole removed is not connected) instead of $\mathbb{C}$ (a two dimensional space with a hole removed is still connected).

6

Let $\mathbb{H}$ denote the skew field of quaternions.

(a) Let G be the set of unit quaternions. Prove that G is a group.

(b) Write and arbitrary quaternion $q = a + b\mathrm{i} + c\mathrm{j} + d\mathrm{k}$ as $q = z + w\mathrm{j}$, where $z = a + b\mathrm{i}$ and $w = c + d\mathrm{i}$ are viewed as complex numbers. Define $F : \mathbb{H} \to \mathcal{M}_2(\mathbb{C})$ by

$$F : z + w\mathrm{j} \mapsto \begin{pmatrix} z & w \\ -\overline{w} & \overline{z} \end{pmatrix}.$$

Prove that F gives a group isomorphism of G onto $\mathsf{SU}(2)$, and that both are homeomorphic to S^3.

(c) Explain why G "agrees" with $\mathsf{Sp}(1) = \mathsf{Sp}(1; \mathbb{C}) \cap \mathsf{U}(2)$ defined in class.

(d) Exhibit a Lie group isomorphism between $\mathsf{SO}(2)$ and $\mathsf{U}(1)$, and prove that both are homeomorphic to S^1.

Solution. (a) The identity is given by $e = 1 \in \mathbb{R}$, inverses are $q^{-1} = a - b\mathrm{i} - c\mathrm{j} - d\mathrm{k}$. To show this group is closed under multiplication please accept my computer aided proof:

```
from sympy.algebras.quaternion import Quaternion
from sympy.abc import a, b, c, d, e, f, g, h

q = Quaternion(a, b, c, d)
r = Quaternion(e, f, g, h)

(q * r).norm().expand().collect([a, b, c, d]).simplify()
>>> sqrt((a**2 + b**2 + c**2 + d**2)*(e**2 + f**2 + g**2 + h**2))
```

Now because both `q` and `r` are unit quaternions (something I wasn't able to tell `sympy`), we know $a^2 + b^2 + c^2 + d^2 = 1 = e^2 + f^2 + g^2 + h^2$. Hence the product also has norm 1.

(b) Let $q = a + b\mathrm{i} + c\mathrm{j} + d\mathrm{k}$ and $r = e + f\mathrm{i} + g\mathrm{j} + h\mathrm{k}$ and note that

$$\begin{aligned} q \cdot r = ae - bf - cg - dh &+ (af + be + ch - dg)\mathrm{i} \\ &+ (ag - bh + ce + df)\mathrm{j} + (ah + bg - cf + de)\mathrm{k} \end{aligned}$$

Then we have the following very fun function.

$$F(q \cdot r) = \begin{pmatrix} ae - bf - cg - dh + (af + be + ch - dg)\mathrm{i} & ag - bh + ce + df + (ah + bg - cf + de)\mathrm{i} \\ -ag + bh - ce - df + (ah + bg - cf + de)\mathrm{i} & ae - bf - cg - dh - (af + be + ch - dg)\mathrm{i} \end{pmatrix}$$

Now we can try the same taking the product after.

$$\begin{aligned} F(q) \cdot F(r) &= \begin{pmatrix} a + b\mathrm{i} & c + d\mathrm{i} \\ -c + d\mathrm{i} & a - b\mathrm{i} \end{pmatrix} \begin{pmatrix} e + f\mathrm{i} & g + h\mathrm{i} \\ -g + h\mathrm{i} & e - f\mathrm{i} \end{pmatrix} \\ &= \begin{pmatrix} ae - bf - cg - dh + (af + be + ch - dg)\mathrm{i} & ag - bh + ce + df + (ah + bg - cf + de)\mathrm{i} \\ -ag + bh - ce - df + (ah + bg - cf + de)\mathrm{i} & ae - bf - cg - dh - (af + be + ch - dg)\mathrm{i} \end{pmatrix} \end{aligned}$$

As you can (hopefully) see, these two are equivalent. Hey I'm not the one who suggested this problem...

Apparently that wasn't enough torture for this problem. To show this is a surjection you can write out the condition $AA^\dagger = \mathbb{1}$ for 2×2 complex matrices, along with the fact that $\det A = 1$ to arrive at the conclusion that any matrix in $\mathsf{SU}(2)$ can be written as $\left(\begin{smallmatrix} z & w \\ -\overline{w} & \overline{z} \end{smallmatrix}\right)$. This argument can also be made because $AA^\dagger = \mathbb{1}$ which means A has orthonormal columns. If the first column is (a, b), and the second must be orthogonal to that. Together with the fact that the determinant of A must be 1 gives us the second column must be $(-b, a)^\dagger$.

These are both homeomorphic to S^3 by sending a unit quaternion $q = a + b\mathrm{i} + c\mathrm{j} + d\mathrm{k}$ to $(a, b, c, d) \in \mathbb{R}^4 \supset S^3$. This is clearly a homeomorphism. I normally wouldn't be so hand wavey[1], but this problem is *way* tedious.

(c) We've just shown G to be isomorphic to $\mathsf{SU}(2)$, so clearly they're in $\mathsf{U}(2)$. Now we just need to show they're also in $\mathsf{Sp}(1; \mathbb{C})$. I tried showing any element in $\mathsf{SU}(2)$ commutes with $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, but that didn't seem to work. I'm out of ideas, and out of time.

(d) $\mathsf{SO}(2)$ is the matrix Lie group of 2×2 rotation matrices, which can always be specified by an angle $\theta \in [0, 2\pi)$. That is any element $A \in \mathsf{SO}(2)$ can be written as R_θ for some θ as previously stated. With this we define $f : \mathsf{SO}(2) \to \mathsf{U}(1)$ by

$$f : R_\theta \mapsto \mathrm{e}^{\mathrm{i}\theta}.$$

The periodicity of the complex exponential ensure this function is a bijection. To show it's a homomorphism we use the simple geometric fact that rotations about the origin compose by adding the corresponding angles of rotation. That is $R_\alpha \cdot R_\beta = R_{\alpha+\beta}$. Thus $f(R_\alpha R_\beta) = f(R_{\alpha+\beta}) = \mathrm{e}^{\mathrm{i}(\alpha+\beta)}$ and $f(R_\alpha)f(R_\beta) = \mathrm{e}^{\mathrm{i}\alpha}\mathrm{e}^{\mathrm{i}\beta} = \mathrm{e}^{\mathrm{i}(\alpha+\beta)}$.

To show $\mathsf{U}(1)$ is homeomorphic to S^1, take the map $\mathrm{e}^{\mathrm{i}x} \mapsto (\cos x, \sin x)$. For $\mathsf{SO}(2)$ take $R_\theta \mapsto (\cos\theta, \sin\theta)$.

[1] maybe I would

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For $X, Y \in \mathcal{M}_n(\mathbb{C})$, define $F_X(Y) := \partial_t|_{t=0} e^{X+tY}$.

(a) Prove that $F_X : \mathcal{M}_n(\mathbb{C}) \to \mathcal{M}_n(\mathbb{C})$ is linear.

(b) Prove that for all $X, Y \in \mathcal{M}_n(\mathbb{C})$ with $\|Y\| < 1$, there holds

$$\left\| e^{X+Y} - e^X - F_X(Y) \right\| \le C\|Y\|^2 e^{\|X\|},$$

where C is some constant independent of X, Y.

(c) Prove that $\exp : X \mapsto e^X$ defines a continuously differentiable function from $\mathcal{M}_n(\mathbb{C})$ to $\mathcal{M}_n(\mathbb{C})$.

Solution. (a)

$$\begin{aligned} F_X(Y) &= \frac{\partial}{\partial t} \lim_{n\to\infty} \left[e^{X/n} e^{tY/n} \right]^n \Bigg|_{t=0} \\ &= \lim_{n\to\infty} \sum_{m=1}^{n} e^{\frac{m}{n}X} \frac{Y}{n} e^{\frac{n-m}{n}X} \\ &= \left(\lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} e^{\frac{m}{n}X} Y e^{\frac{-m}{n}X} \right) e^X \\ &= \int_0^1 e^{xX} Y e^{(1-x)X} \, dx \end{aligned}$$

From here it's simple to see F_X is linear.

(b)

$$\begin{aligned} e^{X+Y} - e^X - F_X(Y) &= \left(\mathbb{1} + X + Y + \sum_{n=2}^{\infty} \frac{(X+Y)^n}{n!} \right) \\ &\quad - \left(\mathbb{1} + X + \sum_{n=2}^{\infty} \frac{X^n}{n!} \right) \\ &\quad - \left(Y + \frac{\partial}{\partial t} \sum_{n=2}^{\infty} \frac{(X+tY)^n}{n!} \Bigg|_{t=0} \right) \\ &= \sum_{n=2}^{\infty} \frac{1}{n!} \left(X^n - (X+Y)^n - \frac{\partial}{\partial t} (X+tY)^n \Bigg|_{t=0} \right) \end{aligned}$$

Taking the norm of both sides it's clear we can get a factor of $e^{\|X\|}$. To get the $\|Y\|^2$ I think it comes from the fact that there are *never* any Y^n terms for any n coming from the last F_X term. That said I cannot find how to get the two simultaneously. :'(

(c) Remember $\mathcal{M}_n(\mathbb{C}) \cong \mathbb{C}^{n^2}$. Since we defined e^X via a power series, and $(X^m)_{ij}$ is a polynomial in the matrix entries (for all m), surely this function is continuously differentable, and in fact, wouldn't it be infinitely differentiable?

I think this fact can also be seen from the fact that both matrix multiplication and addition are smooth maps, and e^X is a composition of smooth maps, and hence smooth. Perhaps this is not always true though because we have ever increasing number of compositions?

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Prove that for all $X \in \mathcal{M}_n(\mathbb{C})$, we have

$$\lim_{m\to\infty} \left[\mathbb{1} + \frac{X}{m}\right]^m = e^X.$$

Solution. Here we will use the fact that for $\|B\| < 1/2$ we have

$$\log(\mathbb{1} + B) = B + \mathcal{O}\left(\|B\|^2\right).$$

Start by choosing an m large enough so that both $\|X/m\| < 1/2$ and $\|X/m - \mathbb{1}\| < 1$ are satisfied. Then by the above identity we have

$$\log\left(\mathbb{1} + \frac{X}{m}\right) = \frac{X}{m} + \mathcal{O}\left(\frac{\|X\|}{m^2}\right).$$

The second inequality we chose m to satisfy allows us to exponentiate both sides to yield

$$\mathbb{1} + \frac{X}{m} = \exp\left(\frac{X}{m} + \mathcal{O}\left(\frac{\|X\|}{m^2}\right)\right)$$

and, therefore

$$\left(\mathbb{1} + \frac{X}{m}\right)^m = \exp\left(X + \mathcal{O}\left(\frac{\|X\|}{m}\right)\right).$$

Taking the limit, and using the continuity of the exponential we find that

$$\lim_{m\to\infty} \left(\mathbb{1} + \frac{X}{m}\right)^m = e^X.$$

9

Prove that, even when $X, Y \in \mathcal{M}_n(\mathbb{C})$ do not commute, we still have

$$\frac{\partial}{\partial t} \operatorname{tr}\left(e^{X+tY}\right)\Big|_{t=0} = \operatorname{tr}\left(e^X Y\right).$$

Solution. Here we make good use of the Lie product formula.

$$\begin{aligned}
\frac{\partial}{\partial t} \operatorname{tr}\left(e^{X+tY}\right)\Big|_{t=0} &= \frac{\partial}{\partial t} \operatorname{tr}\left[\lim_{n\to\infty} \left(e^{X/n}e^{tY/n}\right)^n\right]\Big|_{t=0} \\
&= \operatorname{tr}\left[\lim_{n\to\infty} n\left(e^{X/n}e^{tY/n}\right)^{n-1} e^{X/n}e^{tY/n}\frac{Y}{n}\right]\Big|_{t=0} \\
&= \operatorname{tr}\left[\lim_{n\to\infty} \left(e^{X/n}e^{tY/n}\right)^n Y\right]\Big|_{t=0} \\
&= \operatorname{tr}\left(e^{X+tY}Y\right)\Big|_{t=0} \\
&= \operatorname{tr}\left(e^X Y\right)
\end{aligned}$$

10

Prove that a compact matrix Lie group has only finitely many connected components.

Solution. Take G to be our compact Lie group. Without loss of generality we will think about G as a closed and bounded subset of $\mathbb{R}^n$ for some n. Now suppose G has an infinute number of connected components. Because each component must has an element with an open neighborhood around it, the volume of each component is $\varepsilon > 0$. However our closed and bounded region of $\mathbb{R}^n$ has finite volume and cannot fit an infinite number of disjoint open balls.