Lie Groups and Lie Algebras Assignment 2

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1

A matrix A is called unipotent if $A = \mathbb{1} + N$ with N nilpotent. Note that $\log A$ is defined whenever A is unipotent since in that case $A - \mathbb{1}$ is nilpotent, and the relevant power series terminates after finitely many terms.

(a) Prove that if A is unipotent, then $\log A$, defined using the power series

$$\log A = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{(A - \mathbb{1})^k}{k},$$

is nilpotent, and that $e^{\log A} = A$.

(b) Prove that if X is nilpotent, then e^X is unipotent, and that $\log(e^X) = X$.

Solution. (a) Given $A = \mathbb{1} + N$ where N is nilpotent we take the index of N to be n. First let's expand the series for $\log A$.

$$\begin{aligned}
\log A &= \sum_{k=1}^{\infty} (-1)^{k+1} \frac{(A - \mathbb{1})^k}{k} \\
&= \sum_{k=1}^{\infty} (-1)^{k+1} \frac{N^k}{k} \\
&= \sum_{k=1}^{n-1} (-1)^{k+1} \frac{N^k}{k} \\
&= N - \frac{N^2}{2} + \frac{N^3}{3} - \cdots + (-1)^n \frac{N^{n-1}}{n-1}
\end{aligned}$$

To see how this expression is nilpotent take $(\log A)^n$. Using the fact that $A^\ell = 0$ for all $\ell \geq n$ we can see all terms will vanish, because their degree will be greater than or equal n.

(b) Again let's expand the series for $\exp(X)$ where the index of X is again n.

$$e^X = \sum_{k=0}^{\infty} \frac{X^k}{k!} = \mathbb{1} + \sum_{k=1}^{n-1} \frac{X^k}{k!} = \mathbb{1} + \underbrace{X + \frac{X^2}{2} + \cdots + \frac{X^{n-1}}{(n-1)!}}_{N}$$

Here we see we can chop off the tail of the exponential and it leaves us with $e^X = \mathbb{1} + N$ where N is nilpotent by the same argument as above. That is take N^n and the fact that $N^\ell = 0$ for all $\ell \geq n$.

2

(a) Classify all the one-dimensional and two-dimensional real abstract Lie algebras up to isomorphism.

(b) Find at least three non-isomorphic three-dimensional real Lie algebras. (*Start by proving that* $\mathsf{su}(2)$ *is not isomorphic to* $\mathsf{sl}(2;\mathbb{R})$.)

(c) Prove that $\mathsf{su}(2)$ is isomorphic to $\mathbb{R}^3$ as real Lie algebras, where the Lie algebra structure on the latter is given by the cross product.

Solution. (a) The only one dimensional real Lie algebra is (up to isomorphism) the real line $\mathbb{R}$ with addition with Lie bracket given as the commutator (and hence always 0).

There are two real Lie algebras. The Abelian one with $[X,Y]=0$ for all $X,Y\in\mathfrak{g}$ which can be represented as $\mathbb{R}^2$ under addition. And second we have a Lie algebra with commutation relation $[X,Y]=X$ where X and Y are a basis for our vector space. Any commutation relation $[X,Y]=\alpha X+\beta Y$ can be shown to be equivalent to the previous commutation relation by a basis transformation, and hence are still isomorphic.

(b) First take $\mathbb{R}^3$ with vector addition. This is an Abelian Lie algebra. The second two will be $\mathsf{su}(2)$ and $\mathsf{sl}(2;\mathbb{R})$ which we will now prove are not isomorphic.

First recall the commutation relations for the standard basis of $\mathsf{sl}(2;\mathbb{R})$.

$$[X_1,X_2]=X_2 \qquad [X_1,X_3]=-X_3 \qquad [X_2,X_3]=2X_1$$

And for $\mathsf{su}(2)$:

$$[Y_i,Y_j]=\epsilon_{ijk}Y_k$$

Now note that $\mathsf{sl}(2;\mathbb{R})$ can have multiple two-dimensional *closed* subspaces, but $\mathsf{su}(2)$ does not. Given an isomorphism must preserve subspaces, there cannot be an isomoprhism.

(c) Using the commutation relations above for $\mathsf{su}(2)$, and those of $\mathbb{R}^3$ with the cross product:

$$[\mathbf{x},\mathbf{y}]=\mathbf{z} \qquad [\mathbf{x},\mathbf{z}]=-\mathbf{y} \qquad [\mathbf{y},\mathbf{z}]=\mathbf{x}$$

Relabeling our basis $\mathbf{x}\mapsto\mathbf{x}_1,\mathbf{y}\mapsto\mathbf{x}_2,\mathbf{z}\mapsto\mathbf{x}_3$ we see we can write these commutation relations as

$$[\mathbf{x}_i,\mathbf{x}_j]=\epsilon_{ijk}\mathbf{x}_k$$

which are exactly that of $\mathsf{su}(2)$. So send $\mathbf{x}_i\mapsto Y_i$ for the isomorphism.

3

Define the *n-th generalized Heisenberg group* by

$$H_n = \{A \in \mathsf{GL}(n; \mathbb{R}) : A - \mathbb{1} \text{ is strictly upper triangular}\}.$$

(a) Prove that the Lie algebra of H_n is

$$\mathfrak{h}_n := \{X \in \mathfrak{gl}(n; \mathbb{R}) : X \text{ is strictly upper triangular}\}.$$

(b) Prove that $\exp : \mathfrak{h}_n \to H_n$ is both injective and surjective.

Solution. (a) First note that H_n is connected for all n. To see this take the following path $p : [0, 1] \to H_n$ that starts at the identity and brings you to any element in H_n:

$$p(t) := \begin{bmatrix} 1 & a_{12}t & a_{13}t & \cdots & a_{1n}t \\ & 1 & a_{23}t & & a_{2n}t \\ & & 1 & & \vdots \\ & & & \ddots & a_{n-1,n}t \\ & & & & 1 \end{bmatrix}$$

Where below the diagonal we have all 0s. Since we now have a path from every element in H_n to the identity, we can differentiate it at 0 to find the tangent space.

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}p(t)\right|_{t=0} = \begin{bmatrix} 0 & a_{12} & a_{13} & \cdots & a_{1n} \\ & 0 & a_{23} & & a_{2n} \\ & & 0 & & \vdots \\ & & & \ddots & a_{n-1,n} \\ & & & & 0 \end{bmatrix}$$

This shows the tangent space at the identity contains strictly upper triangular matrices. To see it contains *all* strictly upper triangular matrices choose a_{ij} to be the matrix you want. Done.

(b) First note that any strictly upper triangular matrix is nilpotent. If A is $n \times n$ and strictly upper triangular, the characteristic polynomial of A is A^n and by Cayley-Hamilton we have $A^n = 0$. Thus A is nilpotent. This means the exponential of any element in $\mathfrak{h}_n$ terminates. For $X \in \mathfrak{h}_n$

$$\mathrm{e}^X = \mathbb{1} + X + \frac{X^2}{2} + \cdots + \frac{X^{n-1}}{(n-1)!}.$$

From here, I think there is some argument to be made how each entry in the matrix is a polynomial that's not always zero so something must happen. I'm not entirely sure. This homework has been so long.

4

Give, with justification, an example showing that the connectedness of G is necessary for each of the following statements.

(a) If G is a connected matrix Lie group with abelian Lie algebra, then G is abelian.

(b) If G is a connected matrix Lie group and $\Phi : G \to G$ is a Lie group homomorphism inducing the identity map on $\mathfrak{g}$, then Φ is the identity map on G.

Solution. (a) Let H be any Abelian Lie group and take the product Lie group $G = H \times S_3$ where S_3 is the (non-Abelian) symmetric group on 3 elements. Since S_3 has 6 elements, the new Lie group G is isomorphic to G^6 and hence is not connected. The Lie algebra of G is exactly that of H's because of the discrete structure of S_3. Because H is Abelian, it's Lie algebra is too. Hence we've found an Abelian Lie algebra that arrised from a non-Abelian Lie group.

(b) Take $G = \mathsf{O}(2)$ and $\Phi(A) = \det(A)A$. This is a homomorphism because $\det(AB) = \det(A)\det(B)$, and it induces $\varphi \equiv \mathrm{id}_{\mathsf{so}(2)}$ because of the fact $\det\left(e^X\right) = e^{\operatorname{tr} X} = e^0 = 1$.

5

Let G be a connected matrix Lie group with Lie algebra $\mathfrak{g}$, and let H be a connected subgroup. Prove that H is normal if and only if the Lie algebra $\mathfrak{h}$ of H is an ideal in $\mathfrak{g}$.

Solution. First let $\mathfrak{h}$ be an ideal in $\mathfrak{g}$. By definition we have $\mathrm{ad}_g(h) \in \mathfrak{h}$ for $g \in \mathfrak{g}$ and $h \in \mathfrak{h}$. Applying the fact that $\mathfrak{h}$ is an ideal repeatedly we can conclude $\mathrm{ad}_g^n(h) \in \mathfrak{h}$ for all $n \in \mathbb{N}$. Then, using the fact that $\mathfrak{h}$ is a subspace, and hence closed we can say $\mathrm{e}^{\mathrm{ad}_g}(h) \in \mathfrak{h}$. Now we can use identity proved in lecture relating ad and Ad to say $\mathrm{Ad}_{\mathrm{e}^g}(h) \in \mathfrak{h}$. Using the definition of Ad this means $\mathrm{e}^g h \mathrm{e}^{-g} \in \mathfrak{h}$. Now let's exponentiate both sides, and notice when we exponentiate $\mathfrak{h}$ we get H.

$$\exp(\mathrm{e}^g h \mathrm{e}^{-g}) = \underbrace{\mathrm{e}^g}_{G} \underbrace{\mathrm{e}^h}_{X} \underbrace{\mathrm{e}^{-g}}_{G^{-1}} \in \mathrm{e}^{\mathfrak{h}}$$

This condition expresses the fact that $\mathrm{e}^{\mathfrak{g}}$ normalizes $\mathrm{e}^{\mathfrak{h}}$. To go the last step we notice that $G = \bigcup_{n\in\mathbb{N}} (\mathrm{e}^{\mathfrak{g}})^n$ and $H = \bigcup_{n\in\mathbb{N}} (\mathrm{e}^{\mathfrak{h}})^n$.

Now to go the other direction take H to be a normal subgroup of G. This means $h \mapsto ghg^{-1}$ always lands in H for any $g \in G$. The derivative of this map is $\mathrm{Ad}_g : \mathfrak{h} \to \mathfrak{h}$. In particular for $X \in \mathfrak{g}$ we have

$$\left.\frac{\mathrm{d}}{\mathrm{d}t} \mathrm{Ad}_{\mathrm{e}^{tX}}(Y)\right|_{t=0} = \left.\mathrm{e}^{tX} Y \mathrm{e}^{-tX}\right|_{t=0} = \mathrm{ad}_X(Y) \in \mathfrak{h}$$

This shows $\mathfrak{h}$ is an ideal.

6

In the notation of the lecture "From Lie algebra homomorphism to Lie group homomorphism (II)", prove that the map Φ is indeed a group homomorphism. *(For $A, B \in G$, take paths $p_A, p_B : [0,1] \to G$ leading from the identity to A, B, respectively. Define a path going from $\mathbb{1}$ to AB by setting*

$$p(t) = \begin{cases} p_A(2t) & t \in [0, 1/2] \\ p_B(2t-1)A & t \in [1/2, 1] \end{cases}$$

Now choose admissible partitions for p_A and p_B and try to construct one for p and compute $\Phi(BA)$.)

Solution. Take $a_0, \dots, a_n$ to be "working" partition for p_A and $b_0, \dots b_m$ to be a "working" partition for p_B. Take the following partition for p_{AB}:

$$\frac{a_0}{2}, \dots, \frac{a_n}{2}, \frac{1}{2} + \frac{b_0}{2}, \dots, \frac{1}{2} + \frac{b_m}{2}$$

Now let's compute $\Phi(BA)$.

$$\begin{aligned}
\Phi(BA) &:= f\left(p(\tfrac{1}{2} + \tfrac{b_m}{2})p(\tfrac{1}{2} + \tfrac{b_{m-1}}{2})^{-1}\right) \cdots f(p(\tfrac{a_1}{2})) \\
&= f\left(p_B(b_m)A(p_B(b_{m-1})A)^{-1}\right) \cdots f(p_A(a_1)) \\
&= f\left(p_B(b_m)p_B(b_{m-1})^{-1}\right) \cdots f(p_A(a_1)) \\
&= \Phi(B)\Phi(A)
\end{aligned}$$

7

Let $X \in \mathcal{M}_n(\mathbb{C})$ be a diagonalizable matrix. Prove that ad_X is a diagonalizable linear operator on $\mathcal{M}_n(\mathbb{C})$. How are the eigenvalues of ad_X related to those of X?

Solution. We'll use the following equation to denote the matrix X's eigenvectors and eigenvalues.

$$X\mathbf{x}_i = x_i\mathbf{x}_i$$

Similarly we have

$$X^\dagger \mathbf{y}_i = \overline{x_i}\mathbf{y}_i$$

for another set of eigenvectors of $X^\intercal$. Because they both form a basis for $\mathbb{C}^n$, the product basis $\mathbf{x}_i\mathbf{y}_j^\dagger$ forms a basis for $\mathcal{M}_n(\mathbb{C})$. We can then calculate the effect of ad_X on each one of these basis elements.

$$\begin{aligned}
\mathrm{ad}_X(\mathbf{x}_i\mathbf{y}_j^\dagger) &= X\mathbf{x}_i\mathbf{y}_j^\dagger - \mathbf{x}_i\mathbf{y}_j^\dagger X \\
&= x_i\mathbf{x}_i\mathbf{y}_j^\dagger - \overline{x_j}\mathbf{x}_i\mathbf{y}^\dagger \\
&= (x_i - \overline{x_j})\mathbf{x}_i\mathbf{y}_j^\dagger
\end{aligned}$$

This shows that these basis elements $\mathbf{x}_i\mathbf{y}_j$ are all eigenvectors of ad_X, and hence ad_X can be diagonalized with respect to it.

8

Compute $\log(e^X e^Y)$ directly using the power series for the exponential and logarithm, and verify that the first few terms are given by

$$\log\left(e^X e^Y\right) = X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y]] - \frac{1}{12}[Y, [X, Y]] + \dots$$

Solution. First let's compute the first few terms of $e^X e^Y$.

$$\begin{aligned} e^X e^Y &= \left(\mathbb{1} + X + \frac{X^2}{2} + \frac{X^3}{3!} + \cdots\right)\left(\mathbb{1} + Y + \frac{Y^2}{2} + \frac{Y^3}{3!} + \cdots\right) \\ &= \mathbb{1} + Y + \frac{Y^2}{2} + \frac{Y^3}{3!} + XY + \frac{XY^2}{2} + X + \frac{X^2}{2} + \frac{X^3}{3!} + \cdots \end{aligned}$$

Now we can plug this into the power series for $\log e^X e^Y$ and take all the terms of order 3.

$$\begin{aligned} \log\left(e^X e^Y\right) &= Y + \frac{Y^2}{2} + \frac{Y^3}{3!} + XY + \frac{XY^2}{2} + \frac{X^2 Y}{2} + X + \frac{X^2}{2} + \frac{X^3}{3!} \\ &\quad - \frac{1}{2}\left(Y^2 + Y^3 + YX + YXY + \frac{YX^2}{2} + \frac{Y^2 X}{2} + XY \right. \\ &\quad \left. + \frac{XY^2}{2} + X^2 + X^2 Y + X^3 + XY^2 + XYX\right) \\ &= X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y]] + \frac{1}{12}[Y, [Y, X]] + \cdots \end{aligned}$$

How lovely.

9

Let G be a matrix Lie group with Lie algebra $\mathfrak{g}$.

(a) For a subset $K \subset G$, define the centralizer of K in G to be

$$Z_G(K) = \{g \in G \mid gk = kg \text{ for all } k \in K\},$$

and define the centralizer of K in $\mathfrak{g}$ to be

$$\mathfrak{z}_{\mathfrak{g}}(K) = \{X \in \mathfrak{g} : \mathrm{Ad}_k(X) = X \text{ for all } k \in K\}.$$

Prove that $Z_G(K)$ is a subgroup of G, that $\mathfrak{z}_{\mathfrak{g}}(K)$ a subalgebra of $\mathfrak{g}$, and that $\mathfrak{z}_{\mathfrak{g}}(K)$ is the Lie algebra of $Z_G(K)$.

(b) For a subset $\mathfrak{k} \subset \mathfrak{g}$, define the centralizer of $\mathfrak{k}$ in G to be

$$Z_G(\mathfrak{k}) = \{g \in G \mid \mathrm{Ad}_g(X) = X \text{ for all } X \in \mathfrak{k}\},$$

and define the centralizer of K in $\mathfrak{g}$ to be

$$\mathfrak{z}_{\mathfrak{g}}(\mathfrak{k}) = \{X \in \mathfrak{g} \mid [X, Y] = 0 \text{ for all } Y \in \mathfrak{k}\}.$$

Prove that $Z_G(\mathfrak{k})$ is a subgroup of G, that $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{k})$ a subalgebra of $\mathfrak{g}$, and that $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{k})$ is the Lie algebra of $Z_G(\mathfrak{k})$.

(c) Let H be an analytic subgroup of G, with $\mathfrak{h}$ its Lie algebra. Prove that $Z_G(H) = Z_G(\mathfrak{h})$ and $\mathfrak{z}_{\mathfrak{g}}(H) = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{h})$.

Solution. (a) First we will prove $Z_G(K)$ is a subgroup of G. Take two elements $g, h \in Z_G(K)$. Then

$$(gh)k = g(hk) = g(kh) = (gk)h = (kg)h = k(gh)$$

so it's closed under products. The identity is clearly in $Z_G(K)$ because it commutes with everything. Inverses are in this group because we can multiply on both sides of $gk = kg$ by g^{-1} on the left and right to obtain $kg^{-1} = g^{-1}k$. Hence $Z_G(K)$ is a subgroup.

To show $\mathfrak{z}_{\mathfrak{g}}(K)$ is a subalgebra, take $X, Y \in \mathfrak{g}$ and we'll show $XY \in \mathfrak{g}$.

$$\mathrm{Ad}_k(XY) = kXYk^{-1} = kXk^{-1}kYk^{-1} = \mathrm{Ad}_k(X)\,\mathrm{Ad}_k(Y) = XY$$

Now we can show this subspace is closed under brackets.

$$\mathrm{Ad}_k([X, Y]) = kXYk^{-1} - kYXk^{-1} = XY - YX = [X, Y]$$

So $\mathfrak{z}_{\mathfrak{g}}(K)$ is indeed a subalgebra.

To show the $\mathfrak{z}_{\mathfrak{g}}(K)$ is the Lie algebra of $Z_G(K)$, take $X \in \mathfrak{z}_{\mathfrak{g}}(K)$. By definition we have $ke^{tX}k^{-1} = e^{tX}$ for all $t \in \mathbb{R}$ and $k \in K$. Taking the derivative at 0 on both sides we obtain $kXk^{-1} = X$ which can be rewritten as $kX = Xk$.

To go the other way, let Y be a matrix such that $e^{tY} \in Z_G(K)$ for all t. Then we have $e^{tY}k = ke^{tY}$ for all $k \in K$. The way we defined $\mathfrak{z}_{\mathfrak{g}}(K)$ implies K also contains inverses, so multiply both sides on the right by k^{-1}, and hence the Lie algebra of $Z_G(K)$ is indeed $\mathfrak{z}_{\mathfrak{g}}(K)$.

(b) First we'll show $Z_G(\mathfrak{k})$ is a subgroup of G. The identity element is clearly within the subgroup because it commutes with everything and is it's own inverse. Inverses are within the subgroup because we can multiply both sides of $\mathrm{Ad}_g(X) = gXg^{-1} = X$ on right by g and on the left by g^{-1} to give $X = g^{-1}Xg$. Now take $h = g^{-1}$ and it's easy to see h is in the group. Finally to show closure under products take $g, h \in Z_G(\mathfrak{k})$. Then $\mathrm{Ad}_{gh}(X) = ghXh^{-1}g^{-1} = gXg^{-1} = X$. Hence a subgroup.

Now we'll show $\mathfrak{z}_\mathfrak{g}(\mathfrak{k})$ is a subalgebra. If $X, Y \in \mathfrak{z}_\mathfrak{g}(\mathfrak{k})$, then X and Y commute with everything, and in particular with each other: $[X, Y] = 0$. Hence if Z is any element in the subalgebra we automatically have $[[X, Y], Z] = [0, Z] = 0$. Hence this is a subalgebra.

Finally we show there is a Lie correspondence between these two spaces. Let X be a matrix such that $\mathrm{e}^{tX} \in Z_G(\mathfrak{k})$ for all t. By definition we have $g\mathrm{e}^{tX}g^{-1} = \mathrm{e}^{tX}$ which is equivalent to $g\mathrm{e}^{tX} = \mathrm{e}^{tX}g$ for all $g \in G$. If we restrict g to be elements generated by $k \in \mathfrak{k}$ as e^{tk}, then we can write the above equality as $\mathrm{e}^{tk}\mathrm{e}^{tX} = \mathrm{e}^{tX}\mathrm{e}^{tk}$ which implies k and X commute, and hence in $\mathfrak{z}_\mathfrak{g}(\mathfrak{k})$.

(c) First take $g \in Z_G(H)$ which implies $gh = hg$ for all $h \in H$. The analyticity of H implies $h = \mathrm{e}^{X_1} \cdots \mathrm{e}^{X_n}$ for some $X_i \in \mathfrak{h}$, and hence we can rewrite the commuting condition as

$$g\left(\mathrm{e}^{X_1} \cdots \mathrm{e}^{X_n}\right)g^{-1} = \mathrm{e}^{X_1} \cdots \mathrm{e}^{X_n}$$

And we can expand the exponentials as power series, and push the g's inside each term.

$$\begin{aligned}
g\left(\mathrm{e}^{X_1} \cdots \mathrm{e}^{X_n}\right)g^{-1} &= g\left[\prod_{i=1}^{n}\left(\sum_{n=0}^{\infty}\frac{X_i^n}{n!}\right)\right]g^{-1} \\
&= \prod_{i=1}^{n} g\left(\sum_{n=0}^{\infty}\frac{X_i^n}{n!}\right)g^{-1} \\
&= \prod_{i=1}^{n}\left(\sum_{n=0}^{\infty}\frac{gX_i^n g^{-1}}{n!}\right) \\
&= \prod_{i=1}^{n}\left(\sum_{n=0}^{\infty}\frac{X_i^n}{n!}\right)
\end{aligned}$$

The last equality allows us to imply $gX_ig^{-1} = X_i$ for all i by comparing terms. This shows $Z_G(H) = Z_G(\mathfrak{h})$.

Now for the second equality take $X \in \mathfrak{z}_\mathfrak{g}(H)$. By definition this implies $kXk^{-1} = X$ for all $k \in H$. Again by analyticity of H we can write any k as $\mathrm{e}^{X_1} \cdots \mathrm{e}^{X_n}$. Note that $kXk^{-1} = X$ holds for all k and in particular $k = \mathrm{e}^{X_i}$ for all i. So we have $\mathrm{e}^{X_i}X\mathrm{e}^{-X_i} = X$ and we can exponentiate and move some terms around to obtain $\mathrm{e}^{X_i}\mathrm{e}^{X} = \mathrm{e}X\mathrm{e}^{X_i}$ which implies $[X, X_i] = 0$. Since k was arbitrary and so was X_i, this holds for all k and all X_i. Hence $\mathfrak{z}_\mathfrak{g}(H) = \mathfrak{z}_\mathfrak{g}(\mathfrak{h})$.

10

(a) Prove that every analytic subgroup of $\mathsf{SU}(2)$ is closed. Prove that this is not true for $\mathsf{SU}(3)$.

(b) Let $\mathfrak{a}$ be a subalgebra of the Lie algebra of the Heisenberg group. Prove that $\exp(\mathfrak{a})$ is an analytic subgroup, and is closed.

Solution. (a) Because we've proved $\mathsf{Sp}(1) \cong \mathsf{SU}(2)$ we'll work with the unit quaternions. The Lie algebra of $\mathsf{Sp}(1)$ are the completely imaginary quaternions with no real component. Now take our analytic subgroup $F \subset \mathsf{Sp}(1)$, and $\mathfrak{f}$ it's Lie algebra.

If $\dim \mathfrak{f} = 0$ then $F = \{0\}$ which is clearly closed. If $\dim \mathfrak{f} = 1$, then there is an imaginary quaternion a such that $\mathrm{e}^{k} = \{\cos\theta + a\sin\theta : 0 \leq \theta < 2\pi\}$, and hence is closed. Now when $\dim \mathfrak{f} = 2$ we cannot have a subalgebra, and hence there is no analytic subgroup. To see this recall $x_1 x_2 = -x_2 x_2$[1] if $\{x_1, x_2\}$ is our basis for $\mathfrak{f}$. Hence the commutator $[x_1, x_2] = 2x_1 x_2$ is perpendicular to the basis, and hence the basis does not span the space. Finally if $\dim \mathfrak{f} = 3$, then $k = \mathsf{Sp}(1)$ which is closed.

To see this is not the case for $\mathsf{SU}(3)$ take the following subalgebra of $\mathsf{su}(3)$ where α is irrational.

$$\mathfrak{a} := \left\{ \begin{pmatrix} \mathrm{i}\varphi & 0 & 0 \\ 0 & \mathrm{i}\alpha\varphi & 0 \\ 0 & 0 & -\mathrm{i}\varphi(1+\alpha) \end{pmatrix} : \varphi \in \mathbb{R} \right\}$$

This is exactly the irrational rotations of a torus embedded into $\mathsf{su}(3)$. It's left as an exercise to the reader to verify this has the requisite properties.

(b) First note that for any two elements in the Lie algebra $\mathfrak{h}$ the second order commutators vanish. Hence the Baker-Campbell-Hausdorff formula allows us to write

$$\mathrm{e}^{tX}\mathrm{e}^{tY} = \mathrm{e}^{tX+tY+t^2[X,Y]}$$

Because $\mathfrak{a}$ is a subspace of $\mathfrak{h}$ it is closed under scalar multiplication and addition, and also closed under the Lie bracket. This allows us to conclude $tX + tY + t^2[X,Y] \in \mathfrak{a}$, and hence $\mathrm{e}^{\mathfrak{a}}$ is a subgroup of the Heisenberg group. To prove analyticity we already have $\mathfrak{a}$ is a subalgebra of $\mathfrak{h}$ so we only need to show every element of $\mathrm{e}^{\mathfrak{a}}$ can be written as $\prod \mathrm{e}^{X_i}$, but that's obvious since we're starting with the Lie algebra.

To see $\mathrm{e}^{\mathfrak{a}}$ we'll examine what happens for each possible dimension of $\mathfrak{a}$. First note that for a general element A in $\mathfrak{h}$ we have the following exponential

$$\mathrm{e}^{tA} = \mathbb{1} + tA + \frac{t^2}{2}A^2 = \begin{pmatrix} 1 & ta & tb + \frac{t^2}{2}ac \\ 0 & 1 & tc \\ 0 & 0 & 1 \end{pmatrix}. \tag{1}$$

If $\dim \mathfrak{a} = 1$, then we have control over (say) b and the other two are fixed at 0. Whenever we do this it's clear from eq. (1) that $\mathrm{e}^{\mathfrak{a}}$ is closed. A similar analysis can be done for $\dim \mathfrak{a} = 2$, and for $\dim \mathfrak{a} = 3$ we have the entire Heisenberg algebra.

[1] Because the multiplication of two imaginary quaternions is just like the cross product.