Lie Groups and Lie Algebras Assignment 3

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 # 1

Let G be a matrix Lie group and (Π, V) a representation.

(a) Prove that the representation is irreducible if and only if for all $v \in V \setminus \{0\}$ we have
$$\operatorname{span}_{\mathbb{F}} \{\Pi(A)v : A \in G\} = V,$$
where $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$ according to whether the representation is complex or real.

(b) Prove that the standard representations of $\mathsf{SO}(n)$, $\mathsf{SU}(n)$, $\mathsf{SL}(n;\mathbb{C})$ are irreducible.

Solution. I'll use the notation $\mathbb{F}[Gv]$ to denote $\operatorname{span}_{\mathbb{F}} \{\Pi(A)v : A \in G\}$ which is reminiscent of the notation of a group ring.

(a) $\Longrightarrow$ Take (Π, V) to be irreducible, and suppose $\mathbb{F}[Gv] \neq V$. Then there exists a subspace $W \subseteq V$ not hit by any $\Pi(A)v$ for all $A \in G$. Thus $W^\perp$ is an invariant subspace, and (Π, V) is reducible. By contradiction we're done.

$\Longleftarrow$ Take $\mathbb{F}[Gv] = V$ for all non-zero v and suppose (Π, V) has an irrep $(\Pi|_W, W)$. Then for $w \in W$, then by irreducibility we have $Gw \subseteq W$ and hence $\mathbb{F}[Gw] \subseteq W$. Thus we've found a $v \in V$ such that $\mathbb{F}[Gv] \neq V$ which is a contradiction, and hence (Π, V) must be irreducible.

(b) By (a) if $\mathsf{SO}(n)$, $\mathsf{SU}(n)$, $\mathsf{SL}(n;\mathbb{C})$ were reducible, there would be a vector subspace such that Gv never "hits". Without loss of generality we can take v to be a basis element of $\mathbb{R}^n$ or $\mathbb{C}^n$. Since $\mathsf{SO}(n)$ contains all (orientation preserving) change of bases, it surely contains contains rotating $\mathbf{e}_i$ into $\mathbf{e}_j$ for all i and j. This argument should apply to $\mathsf{SU}(n)$ as well.

To see this is true for $\mathsf{SL}(n;\mathbb{C})$ note that $\mathsf{SU}(n)$ and $\mathsf{SL}(n;\mathbb{C})$ have the same dimension $(n^2 - 1)$. This fact, together with $\mathsf{SU}(n) \subset \mathsf{SL}(n;\mathbb{C})$ and the argument above show the standard representation on $\mathsf{SL}(n;\mathbb{C})$ is irreducible.

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For a smooth function f on $\mathbb{R}^n$ we define $\triangle f := \frac{\partial^2 f}{\partial x_1^2} + \cdots + \frac{\partial^2 f}{\partial x_n^2}$. Prove that for all $A \in \mathrm{O}(n)$ we have $\triangle(f(Ax)) = (\triangle f)(Ax)$.

Solution. We're going to do this by components, so let's recall what it means for A to be orthogonal in components.

$$[AA^\mathsf{T}]_{ij} = \sum_{k=1}^{n} A_{ik}[A^\mathsf{T}]_{kj} = \sum_{k=1}^{n} A_{ik}A_{jk} = \mathrm{col}(i, A) \cdot \mathrm{col}(j, A) = \delta_{ij}$$

Here $\mathrm{col}(i, A)$ denotes the ith column of A.

$$\begin{aligned}
\frac{\partial}{\partial x_i} f(Ax) &= f^{(i)}(Ax) \frac{\partial}{\partial x_i} Ax \\
&= f^{(i)}(Ax)\, \mathrm{col}(i, A) \\
\frac{\partial^2}{\partial x_i^2} f(Ax) &= f^{(ii)}(Ax)\, \mathrm{col}(i, A) \frac{\partial}{\partial x_i} Ax \\
&= f^{(ii)}(Ax)\, \underbrace{\mathrm{col}(i, A) \cdot \mathrm{col}(i, A)}_{1} \\
&= f^{(ii)}(Ax)
\end{aligned}$$

From this we conclude $(\triangle f)(Ax) = \triangle(f(Ax))$.

3

Consider the standard representation of $\mathsf{SO}(2)$ on $\mathbb{R}^2$. Prove that the second statement of Schur's lemma fails. That is, there exists an intertwining map $\mathbb{R}^2 \to \mathbb{R}^2$ which is not a multiple of the identity.

Solution. Recall the standard representation of $\mathsf{SO}(2)$ is the function $\lambda : \mathsf{SO}(2) \to \mathsf{GL}(2;\mathbb{R})$ defined by $\lambda(A)\mathbf{x} := A\mathbf{x}$ for $\mathbf{x} \in \mathbb{R}^2$. Now our goal is to find a function $\psi : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\lambda \circ \psi = \psi \circ \lambda$. Thankfull, 2-dimensional rotations commute, and hence we can pick any $R \in \mathsf{SO}(2)$ to define $\psi_R : \mathbb{R}^2 \to \mathbb{R}^2$ as $\psi_R(\mathbf{x}) = R\mathbf{x}$.

Hence we have

$$\lambda(A)\psi_R(\mathbf{x}) = \lambda(A)R\mathbf{x} = AR\mathbf{x} = RA\mathbf{x} = \psi_R(\lambda(A)\mathbf{x}).$$

4

View the Heisenberg group as sitting in $\mathsf{GL}(3;\mathbb{C})$ and consider the standard representation on $\mathbb{C}^3$. Determine all invariant subspaces. Is this representation completely reducible?

Solution. Let H denote the Heisenberg group and let's run through a computation for the standard representation $\rho : H \to \mathsf{GL}(3;\mathbb{C})$.

$$\begin{aligned}
\rho(h)\mathbf{x} &= h\mathbf{x} \\
&= \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \\
\begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \begin{pmatrix} x + ay + bz \\ y + cz \\ z \end{pmatrix}
\end{aligned}$$

Since this must hold for all $a, b, c \in \mathbb{R}$ we see z and y must be zero. Hence the only invariant subspace is the x-axis.

 # 5

Let $V_n' = \operatorname{span}_{\mathbb{C}}\{z^k : k = 0, \dots, n,\}$ be the set of polynomials in one complex variable of degree at most n. Define an action of $SU(2)$ on V_n' by letting

$$[\Pi(A)f](z) = (-bz + a)^n f\left(\frac{\overline{a}z + \overline{b}}{-bz + a}\right), \text{ for } A = \begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix}.$$

(a) Prove that $\Pi(A)$ does map V_n' to itself for all $A \in \mathsf{SU}(2)$, and that (Π, V_n') is indeed a representation of $\mathsf{SU}(2)$.

(b) Prove that V_n' is isomorphic to $V_n(\mathbb{C}^2)$ as a representation of $\mathsf{SU}(2)$.

Solution. (a)

$$\begin{aligned}[\Pi(A) \cdot f]\,(z) &= (-bz + a)^n \sum_{m=0}^{n} \alpha_m \left(\frac{\overline{a}z + \overline{b}}{-bz + a}\right)^m \\ &= \sum_{m=0}^{n} \alpha_m \left(\overline{a}z + \overline{b}\right)^m (-bz + a)^{n-m} \in V_n'\end{aligned}$$

Now let $AB = \begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix}\begin{pmatrix} c & -\overline{d} \\ d & \overline{c} \end{pmatrix} = \begin{pmatrix} ac - \overline{b}d & -a\overline{d} - \overline{b}\overline{c} \\ bc + \overline{a}d & -b\overline{d} + \overline{a}\overline{c} \end{pmatrix}$ and we'll go through a *very* tedious calculation.

Upon second thought I only have a finite amount of time to work on this and I think I'll learn more from other parts.

(b) Take the following intertwining map $\psi : V_n(\mathbb{C}^2) \to V_n'$ defined by

$$\sum_{m=0}^{n} \alpha_m z_1^m z_2^{n-m} \xmapsto{\ \psi\ } \sum_{m=0}^{n} \alpha_m z_1^m.$$

In short, we're simply dropping the second complex variable z_2. Now we'll show it's actually an intertwining map. Recall we have $(\Pi_n, V_n(\mathbb{C}^2))$ as defined in lecture and (Π, V_n') as defined above.

$$\begin{aligned}[\Pi_n(A) \cdot g]\left(\begin{smallmatrix} z_1 \\ z_2 \end{smallmatrix}\right) &= g\left(A^{-1}\left(\begin{smallmatrix} z_1 \\ z_2 \end{smallmatrix}\right)\right) = g\left(\begin{smallmatrix} \overline{a}z_1 + \overline{b}z_2 \\ -bz_1 + az_2 \end{smallmatrix}\right) \\ &= \sum_{m=0}^{n} \alpha_m \left(\overline{a}z_1 + \overline{b}z_2\right)^m (-bz_1 + az_2)^{n-m} \\ &= \sum_{m,k,\ell=0}^{n,m,n-m} \alpha_m \binom{m}{k}\binom{n-m}{\ell} \overline{a}^k (-b)^\ell \overline{b}^{m-k} a^{n-m-\ell} z_1^{k+\ell} z_2^{n-k-\ell}\end{aligned}$$

And now applying out intertwining map we get

$$\psi\left([\Pi_n(A) \cdot g]\left(\begin{smallmatrix} z_1 \\ z_2 \end{smallmatrix}\right)\right) = \sum_{m,k,\ell=0}^{n,m,n-m} \alpha_m \binom{m}{k}\binom{n-m}{\ell} \overline{a}^k (-b)^\ell \overline{b}^{m-k} a^{n-m-\ell} z_1^{k+\ell}$$

Now let's compute $\Pi(A) \circ \psi$.

$$
\begin{aligned}
\psi\left(g\left(\begin{smallmatrix} z_1 \\ z_2 \end{smallmatrix}\right)\right) &= \sum_{m=0}^{n} \alpha_m z_1^m \\
\Pi(A)\left(\psi\left(g\left(\begin{smallmatrix} z_1 \\ z_2 \end{smallmatrix}\right)\right)\right) &= \sum_{m=0}^{n} \alpha_m \left(\overline{a} z_1 + \overline{b}\right)^m (-b z_1 + a)^{n-m} \\
&= \sum_{m,k,\ell=0}^{n,m,n-m} \alpha_m \binom{m}{k} \binom{n-m}{\ell} \overline{a}^k (-b)^\ell \overline{b}^{m-k} a^{n-m-\ell} z_1^{k+\ell}
\end{aligned}
$$

This is exactly what we got before, so we conclude $\Pi(A) \circ \psi = \psi \circ \Pi_n(A)$.

To conclude ψ is a bijection we use the fact that $f \in V_n'$ is completely characterized by it's coefficients α_i, and this map preserves the number of available coeffiecients. idk how to argue this, but it's pretty obvious in my opinion this is a bijection.

 # 6

Some applications of Schur's lemma. Let V be a complex representation of a compact matrix Lie group G.

(a) Suppose V is equipped with a G-invariant inner product $(\cdot, \cdot)$. Let V_1 and V_2 be irreducible subrepresentations which are non-isomorphic. Prove that $V_1 \perp V_2$ with respect to $(\cdot, \cdot)$.

(b) Prove that, up to multiplication by a positive real scalar, there is a unique G-invariant inner product on V.

Solution. (a) Suppose $V_1 \not\perp V_2$. Then there exists a non-zero vector $v \in V_1 \cap V_2 =: W$. By the irreducibilty of V_1 and V_2 we know $Gv \in V_1$ and $Gv \in V_2$, so $Gv \in W$, and $GW \subseteq W$. Hence W is a subrepresentation contradicting the fact that V_1 and V_2 are irreducible. Thus $V_1 \perp V_2$.

I understand now just because $V_1 \not\perp V_2$ does not imply there is a non-zero vector in their intersection. To recover the proof I think we might be able to argue that because G is compact, it's representation is similar to a unitary one where all subrepresentations are orthogonal.

(b) Let $\langle -, - \rangle$ and $(-, -)$ be two inner products on V. Define the two following maps $\rho_{\langle , \rangle} : V \to V^*$ and $\rho_{(,)} : V \to V^*$ (where V^* is the dual space of V) as

$$\rho_{\langle , \rangle}(v) := \langle v, - \rangle$$
$$\rho_{(,)}(v) := (v, -)$$

Also take the dual representation Π^* as

$$\Pi^*(g)\langle v, - \rangle = \langle v, \Pi(g)^\dagger - \rangle$$

We'll now show both ρ are intertwining maps:

$$\begin{aligned}
(\Pi^*(g) \circ \rho)(v) &= \Pi^*(g)(\rho(v)) \\
&= \Pi^*(g)([v, -]) \\
&= [v, \Pi(g)^\dagger -] \\
(\rho \circ \Pi(g))(v) &= \rho(\Pi(g)(v)) \\
&= [\Pi(g)v, -] \\
&= [v, \Pi(g)^\dagger -]
\end{aligned}$$

Where I'm using $[-, -]$ to denote either of our two inner products. Now by Schur's lemma $\rho_{\langle , \rangle} = \lambda \rho_{(,)}$. To show λ must be real and positive, remember that inner products are positive definite, and so $\underbrace{\langle v, v \rangle}_{\in \mathbb{R}_{\geq 0}} = \lambda \underbrace{(v, v)}_{\in R_{\geq 0}}$. Hence $\lambda \geq 0$.

7

This problem concerns the irreducible representations of $\mathsf{U}(1)$.

(a) For $k \in \mathbb{Z}$, define an action of $\mathsf{U}(1)$ on $\mathbb{C}$ by letting $\Pi_k(g)z = g^k z$. Prove that this defines a representation of $\mathsf{U}(1)$.

(b) Prove that every homomorphism $\Pi : \mathsf{U}(1) \to \mathsf{U}(1)$ has the form $\Pi(g) = g^k$ for some $k \in \mathbb{Z}$.

(c) Prove that every irreducible representation of $\mathsf{U}(1)$ is isomorphic to $(\Pi_k, \mathbb{C})$ for some $k \in \mathbb{Z}$.

Solution. (a) First, the fact that Π_k is a group homomorphism:

$$\Pi_k(g_1 g_2) = (g_1 g_2)^k = g_1^k g_2^k = \Pi_k(g_1)\Pi_k(g_2)$$

Now, to show Π_k is continuous let's look at it's kernel. Indeed it's not hard to see $\ker(\Pi_k) = \{e^{i2\pi n} : n \in \mathbb{N}\}$. This is a closed subgroup of $\mathsf{U}(1)$, so Π_k is continuous.

(b) Suppose $\Pi(g) = g^\alpha$ for $\alpha \in \mathbb{R}$. Anything outside of $\mathbb{R}$ might not maintain closure of $\mathsf{U}(1)$ so it's enough to restrict ourselves to $\mathbb{R}$. Write $\alpha = n + d$ where $n \in \mathbb{Z}$ and $d \in [0, 1)$. We can then rewrite $\Pi(g) = g^n g^d$. Unfortunately we cannot continuously define fractional, and irrational powers of $e^{i\theta}$ for all θ continuosly. This leaves us with $\Pi(g) = g^n$.

(c) Let $(\tilde{\Pi}, V)$ be an irreducible representation of $\mathsf{U}(1)$. The fact that $\mathsf{U}(1)$ is commutative implies $\mathsf{GL}(V)$ must be as well:

$$\tilde{\Pi}(a)\tilde{\Pi}(b) = \tilde{\Pi}(ab) = \tilde{\Pi}(ba) = \tilde{\Pi}(b)\tilde{\Pi}(a)$$

The only commutative general linear groups are $\mathsf{GL}(1;\mathbb{R})$ and $\mathsf{GL}(1;\mathbb{C})$ so V must be one of these. If our representation is into $\mathsf{GL}(1;\mathbb{R})$ then it must be of the trivial representation.

If $V = \mathsf{GL}(1;\mathbb{C}) = (\mathbb{C}_{\neq 0}, *)$, then every $\tilde{\Pi}(e^{i\theta}e^{i\varphi}) = \tilde{\Pi}(e^{i\theta})\tilde{\Pi}(e^{i\varphi})$ because $\tilde{\Pi}$ is a group homomorphism. This means $\tilde{\Pi}|_{\mathsf{U}(1)}$ must also be a group homomorphism, and by (b) it must be of the form Π_k.

8

(a) Prove that $\dim_{\mathbb{C}} \mathcal{H}_m(\mathbb{R}^3) = 2m+1$. (*For $f \in V_m(\mathbb{R}^3)$, we may write*

$$f(x) = \sum_{k=0}^{m} \frac{x_1^k}{k!} f_k(x_2, x_3),$$

where f_k is a homogeneous degree $m-k$ polynomial in x_2, x_3. Now use the condition $\triangle f = 0$ to prove that f is completely determined by f_0 and f_1.)

(b) For $\theta \in \mathbb{R}$, define

$$R_\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix},$$

and consider the subgroup $T = \{R_\theta\}_{\theta \in \mathbb{R}}$ of $\mathrm{O}(3)$. Prove that

$$\mathcal{H}_T = \left\{ f \in \mathcal{H}_m(\mathbb{R}^3) : R_\theta \cdot f = f \text{ for all } \theta \in \mathbb{R} \right\}$$

is one-dimensional.

(c) Suppose W is an invariant subspace of $\mathcal{H}_m(\mathbb{R}^3)$ with respect to the $\mathrm{O}(3)$ representation. Prove that W contains an element of $\mathcal{H}_T$. (*Start by proving that there exists $f \in W$ with $f(1,0,0) \neq 0$, and then consider a suitable integral over $\theta \in [0, 2\pi]$.*)

(d) Prove that $\mathcal{H}_m(\mathbb{R}^3)$ is an irreducible representation of $\mathrm{O}(3)$.

Solution. (a) We'll start by showing $f \in V_m(\mathbb{R}^3)$ can be written as in the hint. Let $d = \dim V_m(\mathbb{R}^3)$.

$$\begin{aligned} f(x,y,z) &= \sum_{i=0}^{d} \alpha_i x^{a_i} y^{b_i} z^{c_i} && a_i + b_i + c_i = m \\ &= \sum_{i=0}^{d} \frac{x^{a_i}}{i!} \alpha_i i! y^{b_i} z^{c_i} \\ &= \sum_{a \in \{a_i\}} \frac{x^a}{a!} f_a(y,z) && f_{a_i}(y,z) := \alpha_i i!\, y^{b_i} z^{c_i} \\ &= \sum_{k=0}^{m} \frac{x^k}{k!} f_k(y,z) && \text{relabeling and } x \text{ has } m \text{ distinct powers} \end{aligned}$$

Now we'll calculate $\triangle f$.

$$\begin{aligned} \triangle f &= \sum_{k=2}^{m} \frac{x^{k-2}}{(k-2)!} f_k(y,z) + \sum_{k=0}^{m} \frac{x^k}{k!} \left(f_k^{(yy)} + f_k^{(zz)} \right) \\ &= \sum_{k=0}^{m-2} \frac{x^k}{k!} \left[f_{k+2}(y,z) + f_k^{(yy)} + f_k^{(zz)} \right] + \frac{x^m}{m!} \left(f_0^{(yy)} + f_0^{(zz)} \right) + \frac{x^{m-1}}{(m-1)!} \left(f_1^{(yy)} + f_1^{(zz)} \right) \\ &= \sum_{k=0}^{m-2} \frac{x^k}{k!} \left[f_{k+2}(y,z) + f_k^{(yy)} + f_k^{(zz)} \right] \end{aligned}$$

Where the last equality holds because $f_0^{(aa)} = 0$ for $a = y, z$ and similarly for $f_1^{(aa)}$.

Now in order for this equation to be identically 0 for all x, y, z we must have the bracketed term equal to 0. Taking $k = 0$ and $k = 1$ we have

$$f_2(y,z) + f_0^{(yy)} + f_0^{(zz)} = 0$$
$$f_3(y,z) + f_1^{(yy)} + f_1^{(zz)} = 0$$

Hence we can "build" all f_i from f_0 and f_1 recursively. This means for an arbitrary $f \in \mathcal{H}_m(\mathbb{R}^3)$ we have to choose a degree m harmonic homogeneous polynomial *and* a degree $m-1$ harmonic homogeneous polynomial. Choosing the first requires $m+1$ numbers, and the second m, so together we have dimension $2m+1$.

(b) First take $\theta = \pi$. Then

$$(R_{-\pi})\mathbf{x} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ -y \\ -z \end{pmatrix}$$

So our solutions must be invariant under $y \mapsto -y$ and $z \mapsto -z$. This means only one of f_0 or f_1 can be non-zero based on the parity of m. This means our $f_R \in \mathcal{H}_T$ look like

$$f_R = f_0(y,z) \qquad \text{or} \qquad f_R = x f_1(y,z).$$

Now using the fact that f_R is invariant under all θ we should theoretically be able to show there is only one free parameter, but I cannot today.

(c) (d)

9

The action considered in Problem #8 also allows us to view $\mathcal{H}_m(\mathbb{R}^3)$ as a representation of $\mathrm{SO}(3)$. Does the proof outlined in Problem #8 show that this representation is irreducible?

Solution. Yes.

 # 10

By Problem #2, the action $(A \cdot f)(x) = f(A^{-1}x)$ gives rise to representations of $\mathsf{O}(2)$ and $\mathsf{SO}(2)$ on $\mathcal{H}_m(\mathbb{R}^2)$.

(a) Prove that $\mathcal{H}_m(\mathbb{R}^2)$ is irreducible as a representation of $\mathsf{O}(2)$.

(b) Prove that $\mathcal{H}_m(\mathbb{R}^2)$ is not irreducible as a representation of $\mathsf{SO}(2)$ for $m \geq 2$.

Solution. (a)

(b)