Lie Groups and Lie Algebras Assignment 4

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 # 1

Let (σ, V) be a complex representation of $\mathsf{sl}(2;\mathbb{C})$. Define H, X, Y as in p.96 of Hall. Let $v \in V \setminus \{0\}$ be an eigenvector of $\sigma(H)$ such that $\sigma(X)v = 0$, and define $v_k = \sigma(Y)^k v$ for $k \geq 0$. Prove that

$$\sigma(X)v_k = k(\lambda - k + 1)v_{k-1}, \text{ for all } k \geq 1.$$

Solution. Let $\sigma(H)v = \lambda v$ and recall the following commutation relations:

$$[X, Y] = H \qquad [H, Y] = -2Y$$

Now let's calculate $\sigma(X)v_k$.

$$\begin{aligned}
\sigma(X)v_k &= \sigma(X)\sigma(Y)^k v \\
&= [\sigma(X)\sigma(Y)]\sigma(Y)^{k-1}v \\
&= [\sigma(H) + \sigma(Y)\sigma(X)]\sigma(Y)^{k-1}v && \sigma(H) = \sigma([X,Y]) = \sigma(X)\sigma(Y) - \sigma(Y)\sigma(X) \\
&\vdots \\
&= k\sigma(H)\sigma(Y)^{k-1}v + \sigma(Y)^k \underbrace{\sigma(X)v}_{0} \\
&= k[\sigma(Y)\sigma(H) - 2\sigma(Y)]\sigma(Y)^{k-2}v && -2\sigma(Y) = \sigma([H,Y]) = \sigma(H)\sigma(Y) - \sigma(Y)\sigma(H) \\
&\vdots \\
&= k\left[\sigma(Y)^{k-1}\underbrace{\sigma(H)v}_{\lambda v} - 2(k-1)\underbrace{\sigma(Y)^{k-1}v}_{v_{k-1}}\right] \\
&= k(\lambda - 2k + 2)v_{k-1}
\end{aligned}$$

Not sure if the question is incorrect or if I missed something. I cannot see it though. I've looked through for at least 2 hours, so if you see my error *please* point it out.

2

Let (Π_1, V_1), (Π_2, V_2) two representations of a connected matrix Lie group. Prove that (Π_1, V_1) is isomorphic to (Π_2, V_2) if and only if (π_1, V_1) is isomorphic to (π_2, V_2), where $(\pi_1, V_1), (\pi_2, V)$ denote the associated Lie algebra representations.

Solution. Suppose $(\Pi_1, V_2) \cong (\Pi_2, V_2)$ by an intertwining map ϕ. Then we have a commutative diagram.

$$\begin{array}{ccc} V_1 & \xrightarrow{\phi} & V_1 \\ {\scriptstyle \Pi_1(g)}\downarrow & & \downarrow{\scriptstyle \Pi_2(g)} \\ V_1 & \xrightarrow{\phi} & V_2 \end{array}$$

Since this holds for all $g \in G$ it will certainly hold for all e^{tX} with $X \in \mathfrak{g}$.

$$\begin{aligned} \left[\phi \circ \Pi_1(e^{tX})\right] v &= \left[\Pi_2(e^{tX}) \circ \phi\right] v \\ \left[\phi \circ e^{t\pi_1(X)}\right] v &= \left[e^{t\pi_2(X)} \circ \phi\right] v \\ \phi\left[e^{t\pi_1(X)} v\right] &= e^{t\pi_2(X)}[\phi(v)] \\ \phi\left[\mathbb{1}v + t\pi_1(X)v + \mathcal{O}(t^2)\right] &= \left[\mathbb{1} + t\pi_2(X) + \mathcal{O}(t^2)\right]\phi(v) \\ \phi(v) + t\phi(\pi_1(X)v) + \mathcal{O}(t^2) &= \phi(v) + t\pi_2(X)\phi(v) + \mathcal{O}(t^2) \\ \phi(\pi_1(X)v) &= \pi_2(X)\phi(v) \end{aligned}$$

Where the last equality is obtained by cancelling $\phi(v)$ on both sides, dividing by t and taking the limit $t \to 0$. This implies $\phi \circ \pi_1 = \pi_2 \circ \phi$.

To go the other way start with $\psi \circ \pi_1(X) = \pi_2(X) \circ \psi$. Any element in a connected Lie group can be written as $g = e^{X_1} e^{X_2} \cdots e^{X_n}$ for some n and some X_i's. Now we'll show $\psi \circ \Pi_1(g) = \Pi_2(g) \circ \psi$.

$$\begin{aligned} \psi\left[\Pi_1(e^{X_1} e^{X_2} \cdots e^{X_n})\right] &= \psi\left[\Pi_1(e^{X_1}) \cdots \Pi_1(e^{X_n})\right] \\ &= \psi\left[e^{\pi_1(X_1)} \cdots e^{\pi_1(X_n)}\right] \\ &= e^{\pi_2(X_1)} \circ \psi \circ e^{\pi_1(X_2)} \circ \cdots \circ e^{\pi_1(X_n)} \\ &= e^{\pi_2(X_1)} \circ \cdots \circ e^{\pi_2(X_n)} \circ \psi \\ &= \Pi_2(e^{X_1} \cdots e^{X_n}) \circ \psi \end{aligned}$$

3

Let V be a real or complex representation of a matrix Lie group or Lie algebra.

(a) Prove that the dual representation V^* is irreducible if and only if V is irreducible.

(b) Prove that $(V^*)^*$ is isomorphic to V as a representation.

(*Given a subspace W of V, its annihilator is the subspace of V^* given by*

$$W^0 = \{l \in V^* : l(w) = 0 \text{ for all } w \in W\}.$$

Recall that $(W^0)^0$ under the canonical vector space isomorphism $V \equiv (V^*)^*$, and thus $W \mapsto W^0$ establishes a one-to-one correspondence between subspaces of V and those of V^*. Look up annihilators if the preceding paragraph is not a review for you.)

Solution. (a) Suppose V^* is an irreducible representation, and let $W \subseteq V$ be an invariant subspace. We can then show W^0 is an invariant subspace of V^* by the following.

$$(\Pi^*(g)l)(w) = l(\Pi(g^{-1})w) = l(\tilde{w}) = 0$$

Where we used the fact that $G \cdot W \subseteq W$ and therefore there must exist a $\tilde{w}$ such that $\tilde{w} = \Pi(g^{-1})w$. Hence W^0 is an invariant subspace of V^*. Since V^* is an irrep, $W^0 = \{\bar{0}\}$ or $W^0 = V^*$.

If $W^0 = V^*$ then *every* linear functional annhilates every vector of W which is only possible when $W = \{0\}$ the zero vector.

If $W^0 = \{\bar{0}\}$ then we want to show $W = V$. We'll do this by contrapositive. So suppose $W \neq V$ is a subspace, and take $\{e_i\}_{i=1}^n$ is a basis for V with W spanned by $\{e_i\}_{i=1}^m$ with $m < n$. Now define the following linear functional

$$f(v) = f\left(\sum_{i=1}^{n} \alpha_i e_i\right) = \sum_{i=m+1}^{n} \alpha_i$$

This is clearly $\mathbb{C}$-linear and homogeneuous, so indeed an element of the dual space. Thus we've found a linear functional such that $f|_W \equiv 0$. This implies $W^0 \neq \{\bar{0}\}$. Thus, by contrapositive, $W = V$.

We've just shown $V^* \text{ irrep} \implies V \text{ irrep}$, and taking the dual of both sides yields $V^{**} \text{ irrep} \implies V^* \text{ irrep}$. Now using the natural isomorphism of vector spaces $V^{**} \cong V$ we have $V \text{ irrep} \implies V^* \text{ irrep}$.

(b) Take the following commutative diagram:

$$\begin{array}{ccc} V & \xrightarrow{\psi} & V^{**} \\ {\scriptstyle \Pi(g)}\downarrow & & \downarrow{\scriptstyle \Pi^{**}(g)} \\ V & \xrightarrow{\psi} & V^{**} \end{array}$$

where $\psi : V \to V^{**}$ is defined as $\psi(v)(\phi) := \text{ev}_v(\phi) = \phi(v)$. To show this commutes we'll have to show $\Pi^{**}(g) \circ \psi = \psi \circ \Pi(g)$. First the left hand side where $v \in V$ and

$f \in V^*$.

$$\begin{aligned}
\Big[(\Pi^{**}(g) \circ \psi)(v)\Big](f) &= \Big[\Pi^{**}(g)(\psi(v))\Big](f) \\
&= \Big[\Pi^{**}(g)(\mathrm{ev}_v)\Big](f) \\
&= \mathrm{ev}_v(\Pi^*(g^{-1})f) \\
&= \Big[\Pi^*(g^{-1})f\Big](v) \\
&= f(\Pi(g)v)
\end{aligned}$$

And now the left hand side:

$$\begin{aligned}
\Big[(\psi \circ \Pi(g))(v)\Big](f) &= \psi(\Pi(g)v)(f) \\
&= \mathrm{ev}_{\Pi(g)v}(f) \\
&= f(\Pi(g)v)
\end{aligned}$$

f and v are completely arbitrary, so this holds for all $v \in V$ and $f \in V^*$. Thus ψ is an intertwining map and we can use Schur's lemma to say ψ is an isomorphism (since it is clearly not 0). Thus $(\Pi, V) \cong (\Pi^{**}, V^{**})$.

4

Let $(\Pi_1, V_1), (\Pi_2, V_2)$ be representations of a matrix Lie group G. Denote by $\mathrm{Hom}(V_1, V_2)$ the space of linear transformations from V_1 to V_2. For $T \in \mathrm{Hom}(V_1, V_2)$ and $g \in G$, define

$$\Pi(g)T = \Pi_2(g) \circ T \circ \Pi_1(g^{-1}).$$

(a) Prove that $(\Pi, \mathrm{Hom}(V_1, V_2))$ is a representation of G.

(b) Prove that $(\Pi, \mathrm{Hom}(V_1, V_2))$ is isomorphic as a representation to $(V_1)^* \otimes V_2$.

(c) Prove that $T \in \mathrm{Hom}(V_1, V_2)$ is an intertwining map with respect to Π_1, Π_2 if and only if $\Pi(g)T = T$ for all $g \in G$.

Solution. (a) Here we will heavily rely on the fact that function composition is associative and we can re-bracket function composition any way we like.

$$\begin{aligned}
\Pi(g_1 g_2)T &:= \Pi_2(g_1 g_2) \circ T \circ \Pi_1\Big((g_1 g_2)^{-1}\Big) \\
&= \Pi_2(g_1 g_2) \circ T \circ \Pi_1\Big(g_2^{-1} g_1^{-1}\Big) \\
&= \Big(\Pi_2(g_1) \circ \Pi_2(g_2)\Big) \circ T \circ \Big(\Pi_1(g_2^{-1}) \circ \Pi_1(g_1^{-1})\Big) \\
&= \Pi_2(g_1) \circ \Big(\Pi_2(g_2) \circ T \circ \Pi_1(g_2^{-1})\Big) \circ \Pi_1(g_1^{-1}) \\
&= \Pi_2(g_1) \circ \Big(\Pi(g_2)T\Big) \circ \Pi_1(g_1^{-1}) \\
&= \Big(\Pi(g_1) \circ \Pi(g_2)\Big) T
\end{aligned}$$

(b) Take the map $\rho : V_1^* \otimes V_2 \to \mathrm{Hom}(V_1, V_2)$ defined by $\rho(f \otimes v)(a) := f(a)v$ where $f \in V_1^*, v \in V_2$ and $a \in V_1$. We'll now show the following diagram commutes.

$$\begin{array}{ccc}
V_1^* \otimes V_2 & \xrightarrow{\ \rho\ } & \mathrm{Hom}(V_1, V_2) \\
{\scriptstyle \Pi_1^*(g) \otimes \Pi_2(g)} \big\downarrow & & \big\downarrow {\scriptstyle \Pi(g)} \\
V_1^* \otimes V_2 & \xrightarrow{\ \rho\ } & \mathrm{Hom}(V_1, V_2)
\end{array}$$

$$\begin{aligned}
\Big(\rho \circ \Big[\Pi_1^*(g) \otimes \Pi_2(g)\Big]\Big)(f \otimes v)(a) &= \rho[\Pi_1^*(g) f \otimes \Pi_2(g) v]\,(a) \\
&= f(g^{-1}a)\Pi_2(g)v \\
\Big(\Big[\Pi(g) \circ \rho\Big](f \otimes v)\Big)(a) &= \Big(\Pi_2(g) \circ f(-)v \circ \Pi_1(g^{-1})\Big)(a) \\
&= \Pi_2(g) v f(g^{-1}a)
\end{aligned}$$

Where I've used the notation $(\Pi_1^*(g)f)(a) = f(g^{-1}a)$ for convenience. Hence ρ is an intertwining map.

To show ρ has an inverse let's look at the function $\tau : \mathrm{Hom}(V_1, V_2) \to V_1^* \otimes V_2$ defined by

$$\tau(\varphi) = \sum_{i=1}^{\dim V_1} e_i^* \otimes \varphi(e_i)$$

where $\{e_i\}$ is a basis for V_1 and $\{e_i^*\}$ is the corresponding dual basis. Now we'll show $\tau \circ \rho = \mathrm{id}_{V_1^* \otimes V_2}$ and $\rho \circ \tau = \mathrm{id}_{\mathrm{Hom}(V_1, V_2)}$.

$$\rho(\tau(\varphi))(v) = \sum e_i^*(v)\varphi(e_i) = \varphi\left(\sum e_i^*(v)e_i\right) = \varphi(v)$$

$$\tau(\rho(f \otimes v)) = \sum e_i^* \otimes \rho(f \otimes v)(e_i) = \sum e_i^* \otimes f(e_i)v = f \otimes v$$

Thus $\tau = \rho^{-1}$ and ρ is an isomorphism.

(c) Take Π_1 to be isomorphic to Π_2 with intertwining map T. This means the following diagram commutes for all $g \in G$.

$$\begin{CD}
V_1 @>T>> V_2 \\
@V{\Pi_1(g)}VV @VV{\Pi_2(g)}V \\
V_1 @>T>> V_2
\end{CD}$$

Or written as an equation we have

$$\Pi_2(g) \circ T = T \circ \Pi_1(g)$$

Now we can compose both sides on the left with $\Pi_1(g^{-1})$.

$$\begin{aligned}
\Pi_2(g) \circ T \circ \Pi_1(g^{-1}) &= T \circ \Pi_1(g) \circ \Pi_1(g^{-1}) \\
&= T \circ \Pi_1(g) \circ \Pi_1(g^{-1}) \\
&= T \circ \Pi_1(gg^{-1}) \\
&= T \circ \Pi_1(e) \\
&= T \circ \mathrm{id}_{V_1} \\
&= T
\end{aligned}$$

This argument used all equivalences, not implications, so this shows the equivalence.

5

Let V be a finite-dimensional real or complex representation of a matrix Lie group or Lie algebra. The following are not necessarily related.

(a) Prove that every non-trivial invariant subspace contains a non-trivial irreducible subrepresentation of V.

(b) Suppose V is irreducible and complex. Consider the direct sum representation $V \oplus V$. Prove that every non-trivial invariant subspace W of V is isomorphic (as a representation) to V, and is of the form

$$W = \{(t_1 v, t_2 v) : v \in V\},$$

for some $t_1, t_2 \in \mathbb{C}$ not both zero.

Solution. (a) Let W be the invariant subspace. When W is one dimensional it's clear that itself is an irreducible subrepresentation. Now assume this is true for $\dim W = n$ and let's look at the case where $\dim W = n + 1$. Write $W = A \oplus B$ where A is one dimensional and B is n-dimensional. There are n ways to do this, but one of them is guaranteed to have an irrep in B.

(b) Please see the next problem to see why any irreducible subrep of $V \oplus V$ is isomorphic to V. To show W is isomorphic to to V, it's clear that it's first a subspace and that $\dim W = \dim V$. By the fact that all finite dimension vector spaces of the same dimension are the same up the isomorphism it is clear that $W \cong V$.

6

Let V_1, V_2 be non-isomorphic, irreducible (real or complex) representations of a matrix Lie group or Lie algebra. Consider the direct sum representation $V_1 \oplus V_2$ and regard V_1, V_2 as subspaces of $V_1 \oplus V_2$ in the obvious way.

(a) Let W be a non-trivial irreducible subrepresentation of $V_1 \oplus V_2$. Prove that $W = V_1$ or V_2.

(b) Prove that V_1, V_2 are the only non-trivial invariant subspaces of $V_1 \oplus V_2$.

Solution. We'll just do part (b) because it implies (a). Let W be a a non-trivial irrep of $V_1 \oplus V_2$ and let $\{\mathbf{e}_i^1\}$ be a basis for V_1 and $\left\{\mathbf{e}_j^2\right\}$ be a basis for V_2 such that W contains some $(\mathbf{e}_i^1, \mathbf{e}_j^2)$ for some particular i and j. By assignment 3 problem 1 we know

$$V = \operatorname{span}_{\mathbb{F}} \{\Pi(A)v : A \in G\}.$$

Let's apply this theorem with $v = (\mathbf{e}_i^1, \mathbf{e}_j^2)$. Thus *any* irrep that contains non-zero vectors in both vector spaces must equal the entire vector space representation. That said if one of the entries in $(-, -)$ is the zero vector, then we can use the following fact

$$[\Pi_1 \oplus \Pi_2(G)](v, 0) = (\Pi_1(G)v, 0).$$

Thus V_1 and V_2 are irreps, and indeed the only ones.

7

Consider the representation $\mathcal{H}_m(\mathbb{R}^3)$ of $\mathsf{SO}(3)$ defined as in the previous assignment, that is, with $\Sigma : \mathsf{SO}(3) \to \mathsf{GL}(\mathcal{H}_m(\mathbb{R}^3))$ given by

$$\Sigma(A)f = f \circ A^{-1}.$$

Denote the associated Lie algebra representation by $\sigma : \mathsf{so}(3) \to \mathsf{gl}(\mathcal{H}_m(\mathbb{R}^3))$ and extend it to $\mathsf{so}(3)_\mathbb{C}$ by complex linearity. Denote the extension by $\widetilde{\sigma}$.

(a) Prove that $\mathsf{so}(3)_\mathbb{C}$ is isomorphic as a complex Lie algebra to $\mathsf{sl}(2;\mathbb{C})$ via

$$\varphi : \begin{pmatrix} 0 & 2a\mathrm{i} & \mathrm{i}(b+c) \\ -2a\mathrm{i} & 0 & c-b \\ -\mathrm{i}(b+c) & b-c & 0 \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & -a \end{pmatrix}.$$

(b) Consider the representation $\widetilde{\sigma} \circ \varphi^{-1}$ of $\mathsf{sl}(2;\mathbb{C})$. Explain how it follows from the previous assignment and what we did in lecture that $\widetilde{\sigma} \circ \varphi^{-1}$ is isomorphic to $(\pi_{2m}, V_{2m}(\mathbb{C}^2))$.

(c) Verify that $h(x,y,z) = (x+\mathrm{i}y)^m$ is a primitive element. That is, prove that $h \in \mathcal{H}_m(\mathbb{R}^3)$, that it is an eigenvector of $\widetilde{\sigma}(\varphi^{-1}(H))$, and that $\widetilde{\sigma}(\varphi^{-1}(X))h = 0$.

(d) Introducing polar coordinates $x = r\sin s\cos t, y = r\sin s \sin t$ and $z = r\cos s$, prove that for $f \in \mathcal{H}_m(\mathbb{R}^3)$ we have

$$\begin{aligned} \widetilde{\sigma}(\varphi^{-1}(H))f &= -2\mathrm{i}\frac{\partial f}{\partial t} \\ \widetilde{\sigma}(\varphi^{-1}(X))f &= \mathrm{e}^{\mathrm{i}t}\left(-\mathrm{i}\frac{\partial f}{\partial s} + \cot(s)\frac{\partial f}{\partial t}\right) \\ \widetilde{\sigma}(\varphi^{-1}(Y))f &= \mathrm{e}^{\mathrm{i}t}\left(\mathrm{i}\frac{\partial f}{\partial s} + \cot(s)\frac{\partial f}{\partial t}\right). \end{aligned}$$

Solution. (a) First recall that $\mathsf{so}(3)_\mathbb{C} \cong \mathsf{so}(n;\mathbb{C})$, which if $a,b,c \in \mathbb{C}$ is spanned by elements of the form in the problem statement above. Please take the following computer assisted proof.

```
from sympy import Symbol, I, simplify
from sympy.matrices import Matrix
from sympy.abc import a, b, c, d, e, f

A = Matrix([
    [             0, 2 * a * I, I * (b + c)],
    [  -2 * a * I,          0,       c - b],
    [-I * (b + c),      b - c,           0]
])
B = Matrix([
    [             0, 2 * d * I, I * (e + f)],
    [  -2 * d * I,          0,       f - e],
    [-I * (e + f),      e - f,           0]
])
```

```
def varphi(mat):
    a = -I * mat[1] / 2
    b = (mat[7] - I * mat[2]) / 2
    c = (-I * mat[2] + mat[5]) / 2
    return Matrix([
        [-a, b],
        [ c, a]
    ])

simplify(varphi(A * B - B * A))
```

$$\begin{bmatrix} bf - ce & -2ae + 2bd \\ 2af - 2cd & -bf + ce \end{bmatrix}$$

```
varphi(A) * varphi(B) - varphi(B) * varphi(A)
```

$$\begin{bmatrix} bf - ce & -2ae + 2bd \\ 2af - 2cd & -bf + ce \end{bmatrix}$$

I love computers.

(b) To show $\tilde{\sigma} \circ \varphi^{-1}$ is isomorphic to $(\pi_{2m}, V_{2m}(\mathbb{C}^2))$

(c) First we show $h \in \mathcal{H}_m(\mathbb{R}^3)$ where $\partial_z^2 h$ is 0.

$$\triangle h = \partial_x^2 h + \partial_y^2 h = m(m-1)(x + \mathrm{i}y)^{m-2} - m(m-1)(x + \mathrm{i}y)^{m-2} = 0$$

Now let's calculate $\tilde{\sigma}$ where $\mathbf{x} = \begin{bmatrix} x & y & z \end{bmatrix}^\intercal$.

$$\begin{aligned} \tilde{\sigma}(X)f &= \frac{\mathrm{d}}{\mathrm{d}t} f(\mathrm{e}^{-tX}\mathbf{x})\bigg|_{t=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}t} f(\mathbf{x}(t))\bigg|_{t=0} \\ &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial t}\bigg|_{t=0} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}\bigg|_{t=0} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t}\bigg|_{t=0} \end{aligned}$$

It's not hard to see that $\frac{\partial \mathbf{x}(t)}{\partial t}\Big|_{t=0} = -X\mathbf{x}$ and hence we have the following equations for the partials:

$$\begin{aligned} \frac{\partial x}{\partial t}\bigg|_{t=0} &= -(X_{11}x + X_{12}y + X_{13}z) \\ \frac{\partial y}{\partial t}\bigg|_{t=0} &= -(X_{21}x + X_{22}y + X_{23}z) \\ \frac{\partial z}{\partial t}\bigg|_{t=0} &= -(X_{31}x + X_{32}y + X_{33}z). \end{aligned}$$

Now we can verify h is a eigenvector of $\tilde{\sigma}(\varphi^{-1}(H))$.

$$\tilde{\sigma}(\varphi^{-1}(H)) = \tilde{\sigma}\left(\begin{bmatrix} 0 & 2\mathrm{i} & 0 \\ -2\mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) = -2\mathrm{i}y\frac{\partial}{\partial x} + 2\mathrm{i}x\frac{\partial}{\partial y}$$

Hence

$$\begin{aligned} \tilde{\sigma}(\varphi^{-1}(H))h &= -2\mathrm{i}ym(x + \mathrm{i}y)^{m-1} - 2xm(x + \mathrm{i}y)^{m-1} \\ &= -2m(x + \mathrm{i}y)^m = h. \end{aligned}$$

And now to show $\tilde{\sigma}(\varphi^{-1}(X))h = 0$.

$$\begin{aligned}\tilde{\sigma}(\varphi^{-1}(X))h &= \left[-\mathrm{i}z\frac{\partial}{\partial x} + z\frac{\partial}{\partial y} + (\mathrm{i}x - y)\frac{\partial}{\partial z}\right](x+\mathrm{i}y)^m \\ &= -\mathrm{i}zm(x+\mathrm{i}y)^{m-1} + \mathrm{i}zm(x+\mathrm{i}y)^{m-1} = 0\end{aligned}$$

(d) To do this problem one must calculate the Jacobian

$$\frac{\partial(x,y,z)}{\partial(r,s,t)}$$

and I do not have time for that today I'm afraid. Once those are calculated you can use the chain rule

$$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial s}{\partial x}\frac{\partial}{\partial s} + \frac{\partial t}{\partial x}\frac{\partial}{\partial t}$$

to calculate the partials and then it's some algebra.

I don't understand what this problem was supposed to show us though, and how it was related to spherical harmonics.

8

Let $\pi : \mathsf{sl}(3;\mathbb{C}) \to \mathsf{gl}(V)$ be an irreducible complex representation of $\mathsf{sl}(3;\mathbb{C})$, and denote by π^* the dual representation, acting on V^*.

(a) Prove that the weights of π^* are the negatives of the weights of π.

(b) Prove that if π has highest weight (m_1, m_2), then π^* has highest weight (m_2, m_1).

Solution. (a) First recall the dual representation of a Lie algebra representation is given by

$$\pi^*(X) = -\pi(X)^\intercal.$$

Also recall the fact that any matrix $A \in \mathcal{M}_n(\mathbb{C})$ satisfies spectrum A = spectrum $A^\intercal$. Thus if

$$\pi(H_1)v = m_1 v \qquad \text{and} \qquad \pi(H_2)v = m_2 v$$

then $m_1 \in \text{spectrum}\,\pi(H_1)^\intercal$ and $m_2 \in \text{spectrum}\,\pi(H_2)^\intercal$. Finally, accounting for the minus sign in the dual representation we have $-m_1 \in \text{spectrum}(-\pi(H_1)^\intercal) = \pi^*(H_1)$ and $-m_2 \in \text{spectrum}(-\pi(H_2)^\intercal) = \pi^*(H_2)$. Thus for any weight (m_1, m_2) belonging to π, $(-m_1, -m_2)$ belongs to π^*.

(b) Let $\mu = (m_1, m_2)$ be the highest weight of π. This means there are $a, b \geq 0$ such that for all weights (m_1', m_2')

$$\begin{aligned} m_1 - m_1' &= 2a - b \\ m_2 - m_2' &= -a + 2b \end{aligned}$$

and thus adding the two equations together we have

$$m_1 + m_2 - (m_1' + m_2') = a + b.$$

Now take $\mu^* = (m_1^*, m_2^*) = (-\hat{m}_1, -\hat{m}_2)$ be the highest weight of π^*. This implies there exist $a', b' \geq 0$ such that for all weights $(m_1^*, m_2^*) = (-\tilde{m}_1, -\tilde{m}_2)$ such that

$$\begin{aligned} \hat{m}_1 - \tilde{m}_1' &= -2a' + b' \\ \hat{m}_2 - \tilde{m}_2' &= a' - 2b' \end{aligned}$$

Again adding the two equations together we have

$$\begin{aligned} \hat{m}_1 + \hat{m}_2 - (\tilde{m}_1 + \tilde{m}_2) &= -(a' + b') \\ \tilde{m}_1 + \tilde{m}_2 - (\hat{m}_1 + \hat{m}_2) &= a' + b' \end{aligned}$$

And this somehow shows $\hat{m}_1 = m_2$ and $\hat{m}_2 = m_1$. Just kidding, I'm pretty lost.