Lie Groups and Lie Algebras Assignment 5

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Due: Wed, Mar 24, 2020 10:00 PM
Course: PMATH 863

1

Consider the adjoint representation of $\mathsf{sl}(3;\mathbb{C})$ as a representation of $\mathsf{sl}(2;\mathbb{C})$ by restriction to the subalgebra $\mathfrak{g}_1 = \operatorname{span}_{\mathbb{C}}\{H_1, X_1, Y_1\} \simeq \mathsf{sl}(2;\mathbb{C})$.

(a) Decompose this representation as a direct sum of irreducible representations of $\mathsf{sl}(2;\mathbb{C})$.

(b) Which isomorphism types appear in the decomposition in (a), and with what multiplicity?

Solution. (a) Since the wording of this question is quite confusing, it's helpful to clarify how I interpreted the question. We're working with the representation $\operatorname{ad}|_{\mathfrak{g}_1} : \mathfrak{g}_1 \to \mathfrak{gl}(\mathsf{sl}(3;\mathbb{C})) = \operatorname{End}(\mathsf{sl}(3;\mathbb{C}))$.

In order to understand the invariant subspaces of $\operatorname{ad}|_{\mathfrak{g}_1}$ we first find the eigenvectors of ad_{H_1} that are annihilated by ad_{X_1}. Indeed the commutation relations easily show we have

$$\begin{aligned} [H_1, X_1] &= 2X_2 & \qquad [X_1, X_1] &= 0 \\ [H_1, Y_2] &= Y_2 & \qquad [X_1, Y_2] &= 0 \\ [H_1, X_3] &= X_3 & \qquad [X_1, X_3] &= 0 \end{aligned}$$

Now we can apply ad_{Y_1} to each one of these eigenvectors to better understand the invariant subspaces. I've ignored constants in the following chains for simplicity.

$$\begin{aligned} &X_1 \xrightarrow{[Y_1,X_1]} H_1 \xrightarrow{[Y_1,H_1]} Y_1 \xrightarrow{[Y_1,Y_1]} 0 \\ &Y_2 \xrightarrow{[Y_1,Y_2]} Y_3 \xrightarrow{[Y_1,Y_3]} 0 \\ &X_3 \xrightarrow{[Y_1,X_3]} X_2 \xrightarrow{[Y_1,X_2]} 0 \end{aligned}$$

Hence we have found 3 invariant subspaces.

(b) Again here is where the wording is very confusing: are we talking about the adjoint representation as a whole, or simply the restriction? All the classmates I talked to thought it was the whole thing. I'll do the whole thing so I don't get points taken off for doing something that wasn't quite asked for, but maybe it was???

Since the adjoint representation is 8 dimensional, and above we found 7, we need one more. Above we never got the vector H_2 so we'll be looking for that. Inspecting the following two commutation relations helps us spot the last:

$$\operatorname{ad}_{Y_1}(H_1) = 2Y_1 \qquad\qquad \operatorname{ad}_{Y_1}(H_2) = -Y_2.$$

Thus the last invariant subspace is spanned by $H_1 + 2H_2$. In sum we have

$$(\operatorname{ad}, \mathsf{sl}(3;\mathbb{C})) \cong (\pi_2, V_2(\mathbb{C}^2)) \oplus (\pi_1, V_1(\mathbb{C}^2)) \oplus (\pi_1, V_1(\mathbb{C}^2)) \oplus (\pi_0, V_0(\mathbb{C}^2))$$

And so the multiplicity of 2 is 1, 1 is 2 and 0 is 1.

2

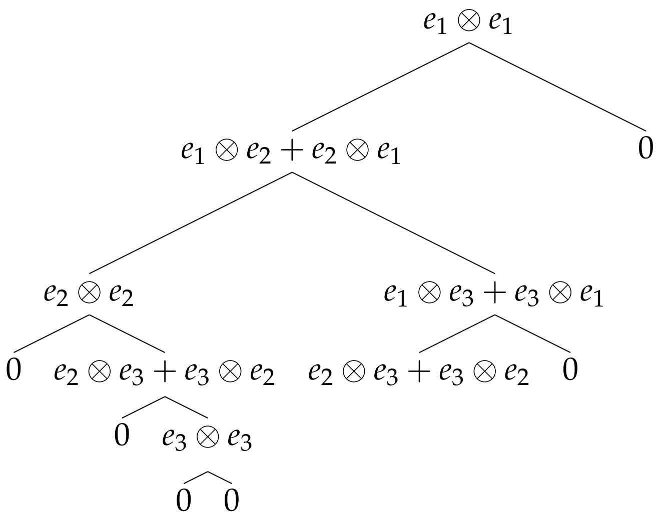
Recall how we constructed an irreducible complex $\mathfrak{sl}(3;\mathbb{C})$ representation with highest weight $(1,1)$ by considering the tensor product representation $\mathbb{C}^3 \otimes (\mathbb{C}^3)^*$.

(a) Use the same method to construct an irreducible complex $\mathfrak{sl}(3;\mathbb{C})$-representation with highest weight $(2,0)$, acting on a subspace of $\mathbb{C}^3 \otimes \mathbb{C}^3$.

(b) Determine the dimension of this representation, along with all the weights and their multiplicities. (The multiplicity of a weight is the dimension of its weight space.)

(c) Decompose $\mathbb{C}^3 \otimes \mathbb{C}^3$, the tensor product of two copies of the standard $\mathfrak{sl}(3;\mathbb{C})$-representation, into a direct sum of irreducible representations.

Solution. (a) Take the product basis of $\mathbb{C}^3 \otimes \mathbb{C}^3$, that is $e_i \otimes e_j$ for $i, j \in \{1,2,3\}$. As a guess we will take $e_1 \otimes e_1$ as the starting point to apply $\pi_{2,0}(Y_1)$ and $\pi_{2,0}(Y_2)$ repeatedly. Branching left indicates Y_1 has been applied, and right indicates Y_2.



Thus we have a 6 dimensional representation spanned by the symmetric vectors of $\mathbb{C}^3 \otimes \mathbb{C}^3$. We can also find the associated weights by adding together the weights of e_i from the standard representation. Inspecting table 1, and calculating $\mu_i - \mu_j = a\alpha_1 + b\alpha_2$

Eigenvector	Weight	Multiplicity
$e_1 \otimes e_1$	$(2,0)$	1
$e_2 \otimes e_2$	$(-2,2)$	1
$e_3 \otimes e_3$	$(0,-2)$	1
$e_1 \otimes e_2 + e_2 \otimes e_1$	$(0,1)$	1
$e_1 \otimes e_3 + e_3 \otimes e_1$	$(1,-1)$	1
$e_2 \otimes e_3 + e_3 \otimes e_2$	$(1,0)$	1

Table 1: Weights of product representation

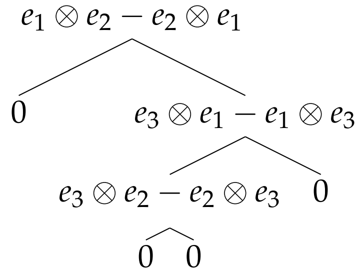
we can (tediously) verify that $(2,0)$ is indeed the highest weight.

(b) Refer to table 1.

(c) Since the above representation is 6 dimensional, and $\mathbb{C}^3 \otimes \mathbb{C}^3$ is 9 dimensional, we need to try and find a representation that lives on the other 3 dimensions. First, note that the "other" 3 dimensions are spanned by

$$e_2 \otimes e_1 - e_1 \otimes e_2 \qquad e_3 \otimes e_1 - e_1 \otimes e_3 \qquad e_3 \otimes e_2 - e_2 \otimes e_3$$

which are the antisymmetric subspace of $\mathbb{C}^3 \otimes \mathbb{C}^3$.[1] To find what this representation looks like we can again apply $\pi_{2,0}(Y_i)$ with the same convention as above.



This tree is exactly that of the standard representation acting on e_1, e_2, e_3, and hence we conclude that we have one copy of the standard representation. In final, we have

$$(\pi_{2,0}, \mathbb{C}^3 \otimes \mathbb{C}^3) \cong (\pi_{1,0} \otimes \pi_{1,0}, \mathrm{Sym}^2(\mathbb{C}^3)) \oplus (\pi_{1,0}, \mathbb{C}^3).$$

Although I'm wondering if that last factor should be $(\pi_{1,0}, \bigwedge^2(\mathbb{C}^3))$? I guess they're isomorphic, so maybe it doesn't matter? Would be good to know.

[1] I wish we learned about symmetric and anti-symmetric powers of vector spaces in this class, because I see them mentioned all over the place when reading material about decomposing representations.

3

Let $V_m(\mathbb{C}^3) = \text{span}_{\mathbb{C}}\left\{z_1^k z_2^l z_3^{m-k-l} : 0 \le k+l \le m\right\}$ and define $(\Pi_m(A)f)(z) = f(A^{-1}z)$ for $A \in \mathsf{SU}(3)$ and $f \in V_m(\mathbb{C}^3)$.

(a) Prove that $(\Pi_m, V_m(\mathbb{C}^3))$ is a complex representation of $\mathsf{SU}(3)$.

(b) Find the weights for π_1 and π_2, the $\mathsf{sl}(3;\mathbb{C})$-representations associated to Π_1 and Π_2, respectively.

(c) Prove that $(\pi_1, V_1(\mathbb{C}^3))$ and $(\pi_2, V_2(\mathbb{C}^3))$ are irreducible representations of $\mathsf{sl}(3;\mathbb{C})$. What are their highest weights?

Solution. (a)

$$\Pi_m(A)\Big(\Big[\Pi_m(B)f\Big]\Big)(z) = \Big[\Pi_m(B)f\Big](A^{-1}z) = f(B^{-1}A^{-1}z) = \Big[\Pi_m(AB)f\Big](z)$$

(b) The action of an arbitrary element $X \in \mathsf{sl}(3;\mathbb{C})$ under the representation π_m is given by

$$\begin{aligned}\pi_m(X) = &-(X_{11}z_1 + X_{12}z_2 + X_{13}z_3)\frac{\partial}{\partial z_1}\\ &-(X_{21}z_1 + X_{22}z_2 + X_{23}z_3)\frac{\partial}{\partial z_2}\\ &-(X_{31}z_1 + X_{32}z_2 + X_{33}x_3)\frac{\partial}{\partial z_3}\end{aligned}$$

Thus, for H_1 and H_2 we have

$$\begin{aligned}\pi_m(H_1) &= z_2\frac{\partial}{\partial z_2} - z_1\frac{\partial}{\partial z_1}\\ \pi_m(H_2) &= z_3\frac{\partial}{\partial z_3} - z_2\frac{\partial}{\partial z_2}\end{aligned}$$

Take $m = 1$ where $V_1 = \text{span}_{\mathbb{C}}\{z_1, z_2, z_3\}$. Applying $\pi_1(H_1)$ and $\pi_1(H_2)$ to an arbitrary element $f = az_1 + bz_2 + cz_3$ and ensuring it is an eigenvector yields the following two equations:

$$\begin{aligned}(m_1+1)az_1 + (m_1-1)bz_2 + cm_1z_3 = 0\\ m_2az_1 + (m_2+1)bz_2 + (m_2-1)cz_3 = 0\end{aligned}$$

From here we can see there are three weights possible tabulated in table 2.

Weight	Eigenvector	Multiplicity
$(1,-1)$	bz_2	1
$(-1,0)$	az_1	1
$(0,1)$	cz_3	1

Table 2: Weight Decomposition for $(\pi_1, V_1(\mathbb{C}^3))$

Take $m = 2$ where $V_2 = \text{span}_{\mathbb{C}} \left\{ z_1^2, z_2^2, z_3^2, z_1z_1, z_1z_3, z_2z_3 \right\}$ and we can repeat the process as above with an arbitrary element $f = az_1^2 + bz_2^2 + cz_3^2 + dz_1z_2 + ez_1z_3 + gz_2z_3$.

$$\pi_2(H_1)f = -2az_1^2 + 2bz_2^2 - ez_1z_3 + gz_2z_3$$
$$\pi_2(H_2)f = -2bz_2^2 + 2cz_3^2 - dz_1z_d + ez_1z_3$$

From here we can read off the weights and eigenvectors, probably much easier than the equation I wrote down for the $m = 1$ case.

Weight	Eigenvector	Multiplicity
$(-2, 0)$	az_1^2	1
$(-2, 2)$	bz_2^2	1
$(0, 2)$	cz_3^2	1
$(0, -1)$	dz_1z_1	1
$(-1, 0)$	ez_1z_3	1
$(1, 0)$	gz_2z_3	1

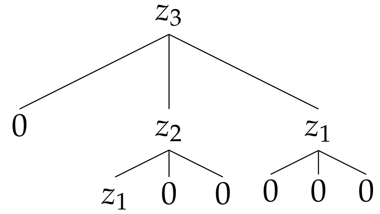
Table 3: Weight Decomposition for $(\pi_2, V_2(\mathbb{C}^3))$

(c) To show $(\pi_1, V_1(\mathbb{C}^3))$ and $(\pi_2, V_2(\mathbb{C}^3))$ are irreps we will first show they are highest weight cyclic representations. Then using Proposition 6.14 from Hall, and the fact that all representations of $\mathfrak{sl}(3; \mathbb{C})$ are completely reducible, we can deduce that the aforementioned representations are irreducible.

For the $m = 1$ case we have highest weight vector $v = cz_3$ with weight $(0, 1)$. This is easily verified (although tedious) by computing $\mu_i - \mu_j = a\alpha_1 + b\alpha_2$ for the weights in table 2. Thus condition 1 is satisfied. Now we can apply each X_i to v to see if it's annihilated.

$$\pi_1(X_1)v = -z_2 \frac{\partial}{\partial z_1}(cz_3) = 0$$
$$\pi_1(X_2)v = -z_3 \frac{\partial}{\partial z_2}(cz_3) = 0$$
$$\pi_1(X_2)v = -z_3 \frac{\partial}{\partial z_1}(cz_3) = 0$$

Thus we also have condition two that $\pi_1(X_i)v = 0$. Lastly we have to verify $V_1(\mathbb{C}^3)$ is the smallest invariant subspace that contains v. We can do this by creating the "tree" applying all $\pi_1(Y_i)$. We use the convention of "left" means apply $\pi_1(Y_1)$, "center" means $\pi_1(Y_2)$ and "right" means $\pi_1(Y_3)$.

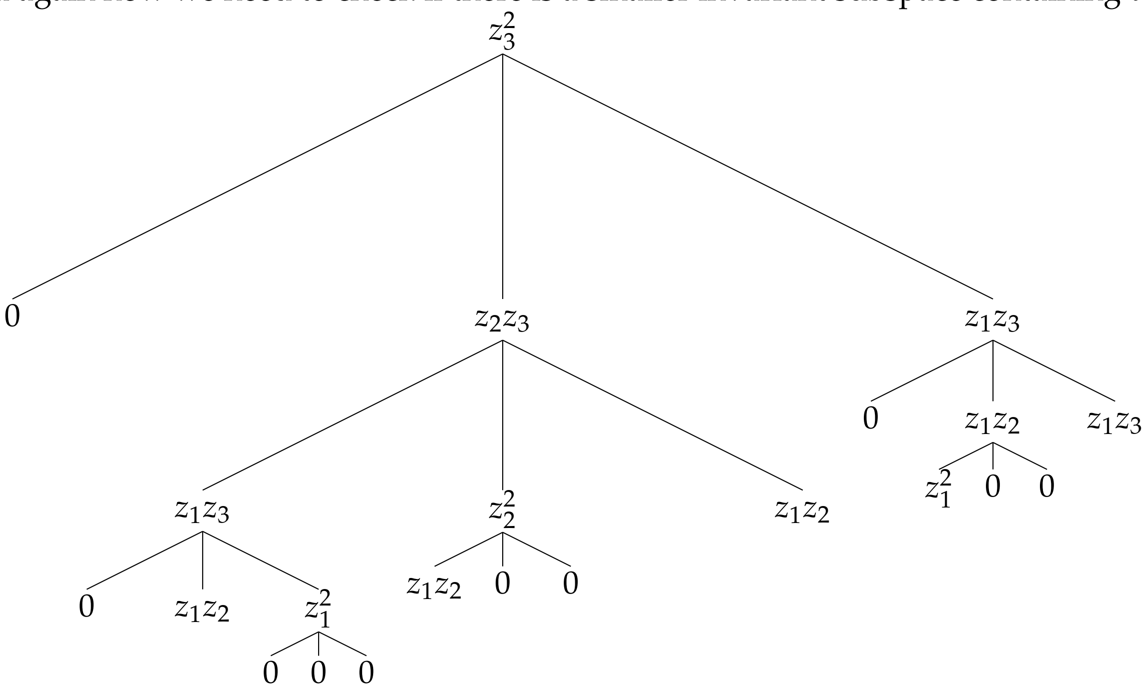


This diagram shows there no invariant subspace containing v that is not the entirety of $V_1(\mathbb{C}^3)$. Thus $(\pi_1, V_1(\mathbb{C}^3))$ is a cyclic representation with highest weight $(0, 1)$ and by the argument given at the outset of (c) we have an irrep.

Now take $m = 2$ and we will run through the same process. The highest weight in table 3 is $(0, 2)$ again by (tediously) computing $\mu_i - \mu_j = a\alpha_1 + b\alpha_2$ repeatedly. We can now check if $v = cz_3^2$ is annihilated by all $\pi_2(X_i)$.

$$\pi_2(X_1)v = -z_2 \frac{\partial}{\partial z_1}\left(cz_3^2\right) = 0$$
$$\pi_2(X_2)v = -z_3 \frac{\partial}{\partial z_2}\left(cz_3^2\right) = 0$$
$$\pi_2(X_2)v = -z_3 \frac{\partial}{\partial z_1}\left(cz_3^2\right) = 0$$

And again now we need to check if there is a smaller invariant subspace containing v.



So indeed this representation is highest weight cyclic with weight $(0, 2)$ and is thus irreducible by the above logic.

4

In each part below, verify that $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g} = \text{Lie}(G)$. Then find the maximal torus in G corresponding to $\mathfrak{t}$.

(a) $G = \mathsf{SO}(2n)$; $\mathfrak{t} = \left\{ \begin{pmatrix} 0 & \theta_1 & & & \\ -\theta_1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & \theta_n \\ & & & -\theta_n & 0 \end{pmatrix} : \theta_i \in \mathbb{R} \right\}$.

(b) $G = \mathsf{SO}(2n+1)$; $\mathfrak{t} = \left\{ \begin{pmatrix} 0 & \theta_1 & & & & \\ -\theta_1 & 0 & & & & \\ & & \ddots & & & \\ & & & 0 & \theta_n & \\ & & & -\theta_n & 0 & \\ & & & & & 0 \end{pmatrix} : \theta_i \in \mathbb{R} \right\}$.

Solution. (a) First lets verify $\mathfrak{t}$ is indeed a Cartan subalgebra. The Lie algebra $\mathsf{so}(2n)$ consists of $2n \times 2n$ skew-symmetric matrices, which clearly $\mathfrak{t}$ is a subset of. In order to show it's a subalgebra, it must be closed under the commutator, but because this is a *Cartan* subalgebra we have the extra condition that $[X, Y] = 0$ for all $X, Y \in \mathfrak{t}$. We'll write elements in $\mathfrak{t}$ in block form using $R_\alpha = \left[\begin{smallmatrix} 0 & \alpha \\ -\alpha & 0 \end{smallmatrix}\right]$.

$$\left[\begin{pmatrix} R_{\theta_1} & & \\ & \ddots & \\ & & R_{\theta_n} \end{pmatrix}, \begin{pmatrix} R_{\phi_1} & & \\ & \ddots & \\ & & R_{\phi_n} \end{pmatrix} \right] = \begin{pmatrix} R_{\theta_1}R_{\phi_1} - R_{\phi_1}R_{\theta_1} & & \\ & \ddots & \\ & & R_{\theta_n}R_{\phi_n} - R_{\phi_n}R_{\theta_n} \end{pmatrix}$$

Now to calculate the terms on the diagonal:

$$\begin{aligned} R_{\theta_i}R_{\phi_i} - R_{\phi_i}R_{\theta_i} &= \begin{pmatrix} 0 & \theta_i \\ -\theta_i & 0 \end{pmatrix}\begin{pmatrix} 0 & \phi_i \\ -\phi_i & 0 \end{pmatrix} - \begin{pmatrix} 0 & \phi_i \\ -\phi_i & 0 \end{pmatrix}\begin{pmatrix} 0 & \theta_i \\ -\theta_i & 0 \end{pmatrix} \\ &= \begin{pmatrix} -\theta_i\phi_i & 0 \\ 0 & -\theta_i\phi_i \end{pmatrix} - \begin{pmatrix} -\theta_i\phi_i & 0 \\ 0 & -\theta_i\phi_i \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

Thus everything in $\mathfrak{t}$ commutes, and is also closed under the bracket/commutator since the zero matrix is skew symmetric.

Now we must show that anything that commutes with *every* element of $\mathfrak{t}$ is also in $\mathfrak{t}$. That is suppose we have some $X \in \mathsf{so}(2n)$ such that $[X, \mathfrak{t}] = 0$. Writing things out in coordinates for $C = XA$ and $D = AX$ we have

$$C_{ij} = \sum_{k=1}^{2n} X_{ik}A_{kj} = X_{i,j+1}A_{j+1,j} = -\theta_j X_{i,j+1}$$
$$D_{ij} = \sum_{k=1}^{2n} A_{ik}X_{kj} = A_{i,i+1}X_{i+1,j} = \theta_j X_{i+1,j}$$

And these must be equal, so we have

$$\theta_i X_{i+1,j} + \theta_j X_{i,j+1} = 0. \tag{1}$$

When $i = j$ then $X_{i+1,i} + X_{i,i+1} = 0$, which A also satisfies. Since eq. (1) must be satisfied for all $X \in \mathfrak{t}$, it must be satisfied for X such that $\theta_i = 0$ for all $i \in \mathbb{Z}_n$ except for one j where $\theta_j = 1$. Plugging these into eq. (1) we see $X_{i,j+1} = 0$ for all $i \neq j$. This, combined with the fact that $X \in \mathsf{so}(2n)$ is anti-symmetric shows that $X \in \mathfrak{t}$.

Now let's compute the maximal torus corresponding to $\mathfrak{t}$. It'll be helpful to compute the first few powers of an element of $\mathfrak{t}$ to get a sense of what's going on.

$$A^2 = \begin{pmatrix} 0 & \theta_1 & & & \\ -\theta_1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & \theta_n \\ & & & -\theta_n & 0 \end{pmatrix}^2 = \begin{pmatrix} -\theta_1^1 & & & & \\ & -\theta_1^2 & & & \\ & & \ddots & & \\ & & & -\theta_n^2 & \\ & & & & -\theta_n^2 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} 0 & \theta_1 & & & \\ -\theta_1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & \theta_n \\ & & & -\theta_n & 0 \end{pmatrix}^3 = \begin{pmatrix} 0 & -\theta_1^3 & & & \\ \theta_1^3 & 0 & & & \\ & & \ddots & & \\ & & & 0 & -\theta_n^3 \\ & & & \theta_n^3 & 0 \end{pmatrix}$$

These give a pretty good hint what the next terms are. Hence we can write

$$\begin{aligned} e^A &= \mathbb{1} + A + A^2 + A^3 + \cdots \\ &= \begin{pmatrix} 1 - \theta_1^2 + \theta_1^4 + \cdots & \theta_1 - \theta_1^3 + \theta_1^5 - \cdots & \\ -\theta_1 + \theta_1^3 - \theta_1^5 + \cdots & 1 - \theta_1^2 + \theta_1^4 + \cdots & \\ & & \ddots \end{pmatrix} \\ &= \begin{pmatrix} \cos\theta_1 & \sin\theta_1 & & & \\ -\sin\theta_1 & \cos\theta_1 & & & \\ & & \ddots & & \\ & & & \cos\theta_n & \sin\theta_n \\ & & & -\sin\theta_n & \cos\theta_n \end{pmatrix} \end{aligned}$$

Thus maximal torus in $\mathsf{SO}(2n)$ is (up to isomorphism) $\operatorname{diag}(R_{\theta_1}, \ldots, R_{\theta_n})$ where $R_\alpha = \left[\begin{smallmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{smallmatrix}\right]$.

(b) The computations performed above are idential for this case, where there is an additional row and column of 0's to work with. Thus the maximal torus of $\mathsf{SO}(2n+1)$ is $\operatorname{diag}(R_{\theta_1}, \ldots, R_{\theta_n}, \mathbb{1}_{2\times 2})$ which is easily seen to be isomorphic to that of $\mathsf{SO}(2n)$. The $\mathbb{1}_{2\times 2}$ arises from the first term of $e^A = \mathbb{1} + \cdots$.

5

(a) Let $n \geq 3$ and let H be the set of diagonal matrices in $\mathrm{SO}(n)$. Prove that H is a maximal closed abelian subgroup of $\mathrm{SO}(n)$, but is not contained in any maximal torus.

(b) By contrast, let H be any closed abelian subgroup of $\mathrm{SU}(n)$. Prove that H is contained in a maximal torus.

Solution. (a) Note that H is the the collection of matrices of the form $\mathrm{diag}(\pm 1, \dots, \pm 1)$ with an even number of -1's on the diagonal. This can be seen from the maximal torus of $\mathrm{SO}(n)$ shown in the previous problem.

First we need to show H is a maximal abelian subgroup of $\mathrm{SO}(n)$. Suppose $A \in \mathrm{SO}(n)$ commutes with all $B \in H$. We can write A in canonical form as

$$A = \mathrm{diag}(R_1, \dots, R_k, \pm 1, \dots, \pm 1)$$

where there are an even number of -1's and 0's everywhere else. Using the fact that commuting matrices preserve each others' eigenspaces we see A must preserve the eigenspaces of B. Since every standard basis vector $\mathbf{e}_i \in \mathbb{R}^n$ is an eigenvector of B, A must map each $A\mathbf{e}_i = \lambda_i \mathbf{e}_i$. Thus all the 2×2 block matrices must be plus or minus 1's. Thus $B \in H$.

(b) Since all $X, Y \in H$ commute, they can be simultaneously diagonalized in some basis. The eigenvalues of a unitary matrix are all unit complex numbers, and hence any element in H can be written as

$$X = \mathrm{diag}(\mathrm{e}^{\mathrm{i}\alpha_1}, \dots, \mathrm{e}^{\mathrm{i}\alpha_n}).$$

Since the determinant of X is 1, we have the condition that $\prod_i \mathrm{e}^{\mathrm{i}\alpha_i} = 1$ which restricts one[2] α_i so we have

$$X = \mathrm{diag}(\mathrm{e}^{\mathrm{i}\alpha_1}, \dots, \mathrm{e}^{-\mathrm{i}\sum_{i=1}^{n-1} \alpha_i}).$$

So every $X \in H$ can be specified by $n - 1$ unit complex numbers. Thus H is clearly contained in the maximal torus of $\mathrm{SU}(n)$.

[2] And can be made to be the last.

6

Let T be the set of diagonal matrices in $\mathsf{U}(n)$ and W its Weyl group. Let S_n be the permutation group of $\{1,\dots,n\}$ and define an action of S_n on T by

$$\sigma \cdot' \operatorname{diag}(u_1,\dots,u_n) = \operatorname{diag}(u_{\sigma^{-1}(1)},\dots,u_{\sigma^{-1}(n)}).$$

(Here we put a prime in the notation to distinguish this action from the action of W on T.) Also, take a generating element $t_0 = \operatorname{diag}(e^{2\pi i\theta_1},\dots,e^{2\pi i\theta_n})$ in T.

(a) Given $w \in W$, prove that there exists a unique $\sigma \in S_n$ such that

$$w \cdot t_0 = \sigma \cdot' t_0.$$

Deduce that $w \cdot t = \sigma \cdot' t$ for all $t \in T$.

(b) In the notation of part (a), prove that the map $w \mapsto \sigma$ defines an injective homomorphism from W into S_n.

(c) Prove that the homomorphism in part (b) is also surjective. (Consequently, W is isomorphic to S_n.)

Solution. (a) We have $w \cdot t_0 = x t_0 x^{-1} = t' \in T$ and because t and t' only differ by conjugation, they must have the same spectrum, however it's possibly "rearranged". This can clearly be done by $\sigma \cdot' t_0$, but we need to show it's unique. Suppose we have $\sigma, \tilde\sigma \in S_n$ such that $w \cdot t_0 = \sigma \cdot' t_0 = \tilde\sigma \cdot' t_0$. Thus we have

$$\operatorname{diag}(e^{2\pi i\theta_{\sigma^{-1}(1)}},\dots,e^{2\pi i\theta_{\sigma^{-1}(n)}}) = \operatorname{diag}(e^{2\pi i\theta_{\tilde\sigma^{-1}(1)}},\dots,e^{2\pi i\theta_{\tilde\sigma^{-1}(n)}})$$

and these must be componentwise equal so

$$e^{2\pi i\theta_{\sigma^{-1}(i)}} = e^{2\pi i\theta_{\tilde\sigma^{-1}(i)}}.$$

This implies $\theta_{\sigma^{-1}(i)} = \theta_{\tilde\sigma^{-1}(i)} + n$ for some $n \in \mathbb{Z}$, but by the linear independence[3] of 1 and the θ_i's, this is only possible if $\sigma^{-1} = \tilde\sigma^{-1}$ and thus $n = 0$, and by the bijectivity of elements in S_n, $\sigma = \tilde\sigma$.

Since t_0 generates, we can always write $t = \lim_{n\to\infty} t_0^{a_n}$ for some subsequence a_n of $\mathbb{Z}$. We then have

$$w \cdot t = x\left[\lim_{n\to\infty} t_0^{a_n}\right]x^{-1} = \lim_{n\to\infty} x t_0^{a_n} x^{-1} = \lim_{n\to\infty}\left[\sigma \cdot' t_0\right]^n = \sigma \cdot' t$$

(b) Let $f : W \to S_n$ be the map such that $w \mapsto \sigma$. This map is indeed a homomorphism:

$$(w_1 w_2) \cdot t = x_1 x_2 t x_2^{-1} x_1^{-1} = x_2(\sigma_2 \cdot' t)x_2^{-1} = (\sigma_1 \circ \sigma_2) \cdot' t$$

To show this map is an injection, suppose $w_1 \mapsto \sigma$ and $w_2 \mapsto \sigma$, that is $f(w_1) = f(w_2)$, where they can only be seen as equal if they are equal on all inputs.

$$\begin{aligned} f(w_1) \cdot' t &= f(w_2) \cdot' t \\ \sigma \cdot' t &= \sigma \cdot' t \\ x_1 t x_1^{-1} &= x_2 t x_2^{-1} \\ \left(x_2^{-1} x_1\right) t &= t\left(x_2^{-1} x_1\right) \end{aligned}$$

[3] over $\mathbb{Q}$

Thus $x_2^{-1}x_1$ commutes with all t, and hence is in T. This implies $x_2^{-1}x_1$ is modded out of the Weyl group and equals the identity in W. Hence $w_1 = w_2$.

(c) To see that f is surjective, note that we can construct a basis change that swaps any basis vectors around in an element x which we conjugate by in $w \cdot t$. Since there are $n!$ ways to swap around basis vectors, we can surely hit every element of S_n.[4]

[4] I know this is sloppy, but I'm *tired*, and I'm not sure what it is, but the Weyl group doesn't *feel* cool.

7

Let G be a compact connected matrix Lie group.

(a) Let $f : G \to H$ be a surjective Lie group homomorphism from G onto another compact connected matrix Lie group. Prove that if T is a maximal torus in G then $f(T)$ is a maximal torus in H. Deduce that if H is abelian then the restriction $f|_T$ is surjective already.

(b) Given $g \in G$ and $n \in \mathbb{N}$, prove that there exists $h \in G$ such that $h^n = g$.

Solution. (a) Since T is connected and compact, and f is a continuous function, $f(T)$ is also connected and compact. Since f is a homomorphism we have $f(a)f(b) = f(ab) = f(ba) = f(b)f(a)$ if $a, b \in T$, and thus $f(T)$ is also commutative and by Theorem 11.2 in Hall, $f(T)$ is a torus.

To show $f(T)$ is maximal in H, take $K \subseteq H$ to be a torus containing $f(T)$. Take an element $h \in K$, and by the surjectivity of f we are guaranteed to be able to find a $g \in G$ such that $f(g) = h$. Since we can write $g = xtx^{-1}$ by Lemma 11.12 in Hall, we have

$$h = f(g) = f(xtx^{-1}) = \underbrace{f(x)}_{\in H}\underbrace{f(t)}_{\in f(T)} f(x)^{-1}.$$

That is, any element $h \in K$ can be decomposed as $h = f(x)f(t)f(x)^{-1}$ which implies $K = yf(T)y^{-1}$, and thus $f(T)$ is maximal.

(b) Since G is compact and connected, we know exp: $\mathfrak{g} \to G$ is surjective, and hence for all $g \in G$ we can fine an $A \in \mathfrak{g}$ such that $g = \mathrm{e}^A$. In particular we can also define $h := \mathrm{e}^{A/n}$ so that $h^n = g$.