Lie Groups and Lie Algebras Assignment 6

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 # 1

Let G by a compact connected matrix Lie group. Given a subset A of G, recall that $Z_G(A) := \{g \in G : gx = gx \text{ for all } x \in A\}$. Also, we write $Z_G(x)$ for $Z_G(\{x\})$.

(a) Prove that every torus is contained in a maximal torus.

(b) Let S be a torus, prove $Z_G(S)$ is the union of all maximal tori in G containing S.

Solution. (a) Let A be a torus in G. If A is maximal, it's contained in itlelf $A \subseteq A$, so we're done. Thus assume A is not maximal. By non-maximality of A there exists a torus T_1 containing it. If it's maximal we're done, so assume it's not and hence $A \subsetneq T_1$. Repeat this argument with T_1 to obtain T_2 and so on. That is we have the following chain of strict inclusions:

$$A \subsetneq T_1 \subsetneq T_2 \subsetneq T_3 \subsetneq \cdots$$

We can now pass to the Lie algebra's where we have

$$\mathfrak{a} \subseteq \mathfrak{t}_1 \subseteq \mathfrak{t}_2 \subseteq \mathfrak{t}_3 \subseteq \cdots \subseteq \mathfrak{g} =: \text{Lie}(G).$$

Since $\mathfrak{g}$ is a *finite* dimensional vector space, we cannot have an infinite chain of strict inclusions, so there must exist an $n \in \mathbb{N}$ such that $\mathfrak{t}_{n+k} = \mathfrak{t}_n$ for all $k \in \mathbb{N}$. However on compacted, connected matrix Lie groups the exponential map is surjective and hence $\exp(\mathfrak{t}_i) = T_i$ and

$$T_{n+k} = \exp(t_{n+k}) = \exp(\mathfrak{t}_n) = T_n$$

but we had $T_n \subsetneq T_{n+k}$ thus we have a contradiction. Hence A is contained in a maximal torus.

(b) Suppose T is a maximal torus containing S. Then by definition we have $T \subseteq Z_G(S)$ and hence $Z_G(S)$ contains all maximal tori containing S, and also their unions.

Now take $g \in Z_G(S)$, or written differently as $S \subseteq Z_G(g)$. Since S is a connected, compact matrix Lie group, so is $Z_G(g)_0$. Take $T \subseteq Z_G(g)_0$ to be a maximal torus that contains S. Since the exponential map is surjective in this case there must exist an element $X \in \text{Lie}(G)$ such that $e^X = g \in Z_G(g)_0$. This implies g is in the center of $Z_G(g)_0$, and using the fact that the $Z(G)$ is equal to the intersection of all maximal tori we conclude $g \in T$. Thus we've found a torus that contains both S and g.

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(a) Let $g \in G$. Prove that $Z_G(g)_0$ is the union of all maximal tori in G containing g.

(b) Specializing to the case $G = \mathrm{SO}(3)$ and let T be the maximal torus corresponding to the Cartain subalgebra given in #4(b) of Assignment 5. Find $g \in \mathrm{SO}(3)$ such that $Z_G(g)_0 = T$ but that $Z_G(g)$ is disconnected.

Solution. (a) See proof to #1(b) to see that $Z_G(g)_0$ is connected.

(b) Take $g = \begin{bmatrix} -1 & & \\ & -1 & \\ & & 1 \end{bmatrix}$.

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Dont think Imma do this one.

Solution.

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Let G be a compact matrix Lie group and V and W irreducible complex representations of G, equipped with G-invariant innter products $(-,-)_V$ and $(-,-)_W$, respectively, which are linear in the first variable and conjugate linear in the second.

(a) Let $\varphi : V \to W$ be an intertwining map. Prove that there exists $\alpha \in \mathbb{R}_{\geq 0}$ such that

$$(\varphi(v), \varphi(v'))_W = \alpha(v, v')_V$$

for all $v, v' \in V$.

(b) Imitate the proof of the orthogonality of characters to prove the following orthogonality relations for matrix coefficients: Given $v_1, v_2 \in V$ and $w_1, w_2 \in W$, there holds

$$\int_G (g \cdot v_1, v_2)_V \overline{(g \cdot w_1, w_2)_W} \, d\mu_G = \frac{(\varphi(v_1), w_1)_W \overline{(\varphi(v_2), w_2)_W}}{\dim V}[V \cong W]$$

where $[A]$ is the Iverson bracket and $\varphi : V \to W$ is any intertwining isometry, that is, and intertwining isomoprhism such that the conclusion of part (a) holds with $\alpha = 1$.

Solution. (a) By Schur's lemma φ is either the 0 map—in which case $\alpha = 0$—or a scalar multiple of the identity. Thus as long as φ is not identically 0, then V and W are isomorphic and by Assignment 3 problem 6 there is only one G-invariant inner product up to a positive constant.

$$(\varphi(v), \varphi(v'))_W = (\beta v, \beta v')_W = |\beta|^2 (v, v')_W = \underbrace{|\beta|^2}_{\geq 0} \underbrace{\gamma}_{\geq 0} (v, v')_V$$

Thus if we take $\alpha := |\beta|^2 \gamma$ then the above equation is satisfied.

(b) Let Π and Σ be the irreps corresponding to V and W respectively. Define the map $L : W \to V$.

$$L := \int_G \Pi(g) \circ \varphi^{-1} \circ \Sigma(g)^\dagger \, d\mu(g)$$

and note that $\Pi(h) \circ L \circ \Sigma(h^{-1}) = L$ by the invariance of the Haar measure and so $\Pi(h) \circ L = L \circ \Sigma(h)$ and hence L is an intertwining map. By Schur's lemma $L = 0$ or $L = \lambda \mathbb{1}$. When $L = 0$ we can take $\varphi^{-1}(w) = (w, w_1)v_1$ and thus

$$\begin{aligned}
0 &= (L(w_2), v_2) \\
&= \int_G (\Pi(x) \circ \varphi^{-1} \circ \Sigma(x)^\dagger w_2, v_2) \, d\mu(x) \\
&= \int_G (\Pi(x)(\Sigma(x^{-1})w_2, w_1)v_1, w_1) \, d\mu(x) \\
&= \int_G (\Pi(x)v_1, v_2)(\Sigma(x^{-1})w_2, w_1) \, d\mu(x) \\
&= \int_G (\Pi(x)v_1, v_2)\overline{(\Sigma(x)w_1, w_2)} \, d\mu(x)
\end{aligned}$$

Now we have the case where $V \cong W$ and φ can be treated as a map $\varphi : V \to V$. As above we have $L = \lambda \mathbb{1}$ and taking the trace of both sides we have $\operatorname{tr}(L) = \lambda \dim V = \operatorname{tr}(\varphi)$. Thus

$$(L(v_2), v_1) = \frac{\operatorname{tr}(\varphi)}{\dim V}\overline{(v_1, v_2)}.$$

Similar to above we take $\varphi(v) = (v, w_2)v_1$. so that

$$\begin{aligned}
\frac{(v_1, w_2)(\overline{(v_2, w_2)})}{\dim V} &= \frac{\operatorname{tr}(\varphi)}{\dim V}\overline{(v_1, w_2)} \\
&= (L(w_2), v_1) \\
&= \int_G (\Pi(x) \circ \varphi \circ \Pi(x^{-1})w_2, v_2)\, \mathrm{d}\mu(x) \\
&= \int_G (\Pi(x)v_1, v_2)(w_2, \Pi(x)w_1)\, \mathrm{d}\mu(x)
\end{aligned}$$

Thus, done. I understand this is probably sloppy.

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Some character computations.

(a) Let χ denote the character of the irreducible representation $\mathcal{H}_m(\mathbb{R}^3)$ of $\mathrm{SO}(3)$. Compute $\chi(g)$ for $g \in \mathrm{SO}(3)$ of the form

$$g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}.$$

(b) Recall the irreducible $\mathrm{SU}(2)$-representations $(\Pi_m, V_m(\mathbb{C}^2))$. Use the character computation to prove that, as representations of $\mathrm{SU}(2)$, we have for non-negative integers $m \geq n$ that

$$V_m \otimes V_n \cong V_{m+n} \oplus V_{m+n-2} \oplus \cdots \oplus V_{m-n+2} \oplus V_{m-n}$$

Solution. (a)

(b) Let $g = \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}$ be an element of the maximal torus of $\mathrm{SU}(2)$.

$$\begin{aligned}
\chi_{V_m \otimes V_n}(g) &= \chi_m(g)\chi_n(g) \\
&= \left(\sum_{k=0}^{m} e^{i(m-2k)\theta}\right)\left(\sum_{j=0}^{n} e^{i(n-2j)\theta}\right) \\
&= \sum_{k,j=0,0}^{m,n} e^{i(m+n-2k-2j)\theta} \\
&= \sum_{l=0}^{n} \sum_{j=p}^{m+n-l} e^{i(m+n-2k)\theta} \\
&= \sum_{l=0}^{n} \sum_{j=0}^{m+n-2l} e^{i(m+n-2l-2k)\theta} \\
&= \sum_{l=0}^{n} \chi_{m+n-2l}(g)
\end{aligned}$$

This shows the representations are equal, and since every element in $\mathrm{SU}(2)$ can be written as $x = yty^{-1}$ with t in the torus, and characters are class functions this must be true on the whole of $\mathrm{SU}(2)$.

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Let G be a compact matrix Lie group.

(a) Let (Π, V) be a complex representation of G and χ it's character. Prove that $|\chi(g)| \leq \dim V$, with equality holding if and only if $\Pi(g)$ is multiplication by a scalar. Here $g \in G$ is an arbitrary element.

(b) Prove that g belongs to $Z(G)$, the center of G, if and only if $|\chi_V(g)| = \dim V$ for every irreducible complex representation V of G. Here χ_V denotes the character of V.

Solution. (a) The compactness of G implies (Π, V) is unitary, and hence $\Pi(g)$ is a normal matrix, with eigenvalues $\mathrm{e}^{\mathrm{i}\theta_i}$ where i ranges from 1 to $k \leq \dim V$. Now since the trace is equal to the sum of the eigenvalues we have

$$|\chi(g)| = \left|\sum_{i=1}^{k} \mathrm{e}^{\mathrm{i}\theta_i}\right| \leq \sum_{i=1}^{k} \left|\mathrm{e}^{\mathrm{i}\theta_i}\right| = \sum_{i=1}^{k} 1 = k \leq \dim V.$$

In the case when $\Pi(g)$ has full rank ($k = \dim V$) then it's not hard to see that $\left|\sum_{i=1}^{\dim V} \mathrm{e}^{\mathrm{i}\theta_i}\right| = \dim V$ implies that all of the θ_i are equal (up to 2π). We can then rewrite all the eigenvalues as $\mathrm{e}^{\mathrm{i}\alpha + \mathrm{i}\tilde{\theta}_i} = \mathrm{e}^{\mathrm{i}\alpha}\mathrm{e}^{\mathrm{i}\tilde{\theta}_i}$. Thus $\Pi(g) = \mathrm{e}^{\mathrm{i}\alpha}\mathbb{1}_V$.

If $\Pi(g) = \beta\mathbb{1}_V$, then since the representation is unitary $\Pi(g)\Pi(g)^\dagger = \mathbb{1}_V$ which implies $\beta = \mathrm{e}^{\mathrm{i}\varphi}$. Thus all of the eigenvalues are $\mathrm{e}^{\mathrm{i}\varphi}$ and since the identity map is full rank $|\chi(g)| = \dim V$.

(b) Suppose $g \in Z(G)$. Then $\Pi(g)$ is an intertwining map and by Schur's lemma $\Pi(g) = \alpha\mathbb{1}$ which implies $|\chi_V(g)| = \dim V$ as above.

Now take $|\chi_V(g)| = \dim V$. As we've shown above $\Pi(g)$ must be a multiple of the identity and hence

$$\Pi(gx) = \Pi(g)\Pi(x) = \alpha\mathbb{1}_V\Pi(x) = \Pi(x)\alpha\mathbb{1}_V = \Pi(x)\Pi(g) = \Pi(xg)$$

Since G is a matrix Lie group Π is a faithful representation or isometrically similar to one, so thus we can conclude $gx = xg$.