Logic and Computability Assignment 1

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Problem 1: ZFC

(a) Let S be a set whose elements are non-empty sets and let T be the set that is the union of the elements of S. Show using the axioms (of ZFC) that there is a map $f : S \to T$ such that $f(s) \in s$ for all $s \in S$. Hint: well-order T. Then for $s \in S$ consider $s \cap T$. Now use the fact that T is well-ordered.

(b) Let X be a partially ordered set with partial order $\prec$. Suppose that all chains in X have an upper bound. Show the existence of a choice function from (a) that the set X has a maximal element with respect to $\preceq$.

Solution. (a) As suggested by the hint, well order T and consider the subset $s \cap T \subseteq T$ for an arbitrary $s \in S$. Since T is well-ordered, this subset must have a least element which we denote by s_{min}. Now define our function $f : S \to T$ by $f(s) := s_{\text{min}}$. By the definition of s_{min}, this is both in T (satisfying the correct range), and in s. Written differently $f(s) := s_{\text{min}} \in s$ for all $s \in S$.

(b) Following the hint, we will proceed by contradiction and assume that all chains in X have an upper bound, but X has not maximal element.

Step 1: Let f be a "choice function"'[1] on non-empty subsets of $\mathcal{P}(X)$ (to X). For a chain T denote by $\text{Upp}(T)$ the set of upper bounds of T (in X) which, by assumption, is non-empty. Let $x = f(\text{Upp}(T))$. We now show that there is a subset $\overline{T} \subset \text{Upp}(T)$ that is non-empty and contains upper bounds that are not contained in T. With $t \in \text{Upp}(T)$ and the non-existence of maximal elements in X, there must exist a $t' \in X$ such that $t \prec t'$. While t could be in T or not, t' is surely not in T as it is strictly "greater than" t, and t is an upper bound. Thus, for every chain T, we have can find a non-empty set $\overline{T}$ that contains strict upper bounds of said chain. Define g to be a function on chains in X that chooses one such strict upper bounds: that is $g(T)$ is an element of $\overline{T}$. As noted in the hint, we can take $g(\varnothing) = t_0$ for some $t_0 \in X$ since $x \prec t_0$ for all $x \in \varnothing$ vacuously.

Step 2: First note that $\{t_0\}$ is indeed a chain: $t_0 \preceq t_0$ and hence all elements are comparable. Substituting $T = \{t_0\}$ in the definition of a nice chain we have

$$g(\{u \in \{t_0\} : u \prec t_0\}) = g(\varnothing) =: t_0.$$

Thus we've found $\{t_0\}$ to be a nice chain, and hence they always exist as long as X is non-empty!

Step 3.a: Let $a \in I_A(x)$ and suppose $a \notin B$. By the definition of $I_A(x)$, $a \in A$, and $a \prec x$, which invalidates the definition of x being the smallest element of $A \setminus B$. Thus $a \in B$, and hence $I_A(x) \subseteq B$.

Step 3.b: First note that $I_B(y) \subsetneq B$ since $y \in B$, and $I_B(y)$ is everything that strictly precedes y. Now when comparing z and x we can compare them based on the sets they are drawn from: $A \setminus I_B(y)$ and $A \setminus B$ respectively. Since B strictly contains $I_B(y)$ we have $A \setminus B \subsetneq A \setminus I_B(y)$. Hence z must be at least as small as x: $z \preceq x$.

[1] I think this refers to the functions we work with in (a), but I don't think we ever defined what a choice function is.

Step 3.c: We begin with $u \in I_B(y)$, $v \in A$, and $v \prec u$. Immediately from $v \prec u$, $u \prec y$ (which comes from $u \in I_B(y)$), and the transitive property of $\prec$ we have $v \prec y$. In order to show $u \in I_A(x)$ we must now show $u \in A$, and $u \prec x$. If we suppose $u \notin A$ then $u \prec y$ would contradict the definition of y being the smallest element in $B \setminus I_A(x)$ and thus $u \in A$. By the definition of x, everything in b precedes it,[2] and hence $u \prec x$ putting $u \in I_A(x)$. Since $v \in A$, $v \prec u$, and $u \in I_A(x)$, we must have $v \in I_A(x)$ as well. Finally by **Step 3.a** $v \in B$, and since $v \prec y$ we also have $v \in I_B(y)$.

Step 3.d: To show $I_A(z) \subseteq I_B(y)$ recall **Step 3.a** where we show $I_A(x) \subseteq B$ combined with **Step 3.b** which shows $z \preceq x$ to combine to say $I_A(z) \subseteq B$. Then we must show that for all $a \in I_A(z)$ we have $a \prec y$. Since $a \in I_A(z)$ we have $a \prec z$, and by $z \preceq x$ we also have $a \prec x$. Finally because y is the least element in $B \setminus I_A(x)$ we must have $a \prec y$ in order to preserve the definition of y being the least element. Thus $I_A(z) \subseteq I_B(y)$. To go the other way take $b \in I_B(y)$. Since $b \prec y$ and y is the least element in $B \setminus I_A(x)$, $b \in A$ and $b \prec x$: that is $b \in I_A(x)$. We also have $b \neq z$ because.... By **Step 3.c** we cannot have $z \prec b$ since then $z \in I_B(y)$ which contradicts the definition of z as being the smallest element *in* $A \setminus I_B(y)$. Since $b \neq z$ and $z \not\prec b$, we must have $z \succ b$. Finally this forces $b \in A$, and hence $b \in I_A(z)$. This means $I_A(z) = I_B(y)$.

Step 3.e: Let us first show that $z = y$ using the fact that A and B are *nice* chains (with respect to g).

$$z = g(\{u \in A : u \prec z\}) = g(I_A(z)) = g(I_B(y)) = g(\{u \in B : u \prec y\}) = y$$

Now we cannot have $z = x$ because if we did, then $y = x$ as well, but $x \in A \setminus B$, whereas $y \in B \setminus I_A(x)$. That is $x \notin B$, and $y \in B$. Thus $z \prec x$ and hence $z \in I_A(x)$, and by equality with z, $y \in I_A(x)$. This contradicts the definition of y being *in* $B \setminus I_A(x)$. This allows us to conclude that $I_A(x) = B$, and hence the definition of y is vacuous as $B \setminus I_A(x) = \emptyset$. Since we assumed $A \setminus B$ is non-empty, we have $B = \{u \in A : u \prec t\}$ for some $t \in A$. The third possibility would hold if we assumed $B \setminus A$ was non-empty.

Step 4: To see that T_∞ is a chain, notice that all $t \in T_\infty$ must have come from a nice chain in $\mathcal{P}(X)$. So there exists a nice chain A containing t. By **Step 3** all nice chains are either equal, or subsets of one another and hence "comparable". Now let $Z \subseteq T_\infty$, $z \in Z$, and let f (our choice function) pick a nice chain containing z, and call it A. The well-ordering of A means $A \cap Z$ has a least element a. To see a is a least element of Z, assume there is a $b \in Z$ with $b \prec a$. Since $b \notin A$, it must be in some other nice chain $B \neq A$. We are now in a situation where A is not an initial segment of B and vice versa, contradiction the claim in **Step 3**. Thus a is the least element of Z and hence T_∞ is well-ordered. Finally we are to prove T_∞ is also nice. Let A be a nice chain with $z \in A$. We claim that

$$\{u \in T_\infty : u \prec z\} = \{u \in A : u \prec z\}.$$

Assume, by way of contradiction that this is not the case: that there is a $y \in T_\infty$ with $y \prec z$ such that $y \notin A$. We can then find another chain B containing y, and by the key claim in **Step 3** we again have a situation where we have two chains that are not proper initial segments of one another. Thus the equality of sets is true. As we've defined g we then have

$$z = g(\{u \in A : u \prec z\}) = g(\{u \in T_\infty : u \prec z\})$$

and thus T_∞ is *niiiiice*.

[2] I'm not actually convinced of this, but I'm not sure what else to argue.

Step 5: Let $v := g(T_\infty)$ and by the definition of g, $v \in \overline{T}$, and hence a strict upper bound for T_∞. We can then see $T_\infty \cup \{v\}$ is a nice chain because it is first a chain with $t \prec v$ for all $t \in T_\infty$, and it is nice because... well I'm not actually sure how we get $g(?) = v$. This is a contradiction because T_∞ was supposed to be the union of all nice chains in $\mathcal{P}(X)$. Thus, our original assumption that X has no maximal elements is false, and X does contain at least one maximal element.

Problem 2: S-formulas

(a) Use induction to show that if S is a first-order alphabet and $\phi, \phi' \in L^S$ then if ϕ is a prefix of ϕ' then $\phi = \phi'$. Show that this is no longer true if we use suffixes instead of prefixes.

(b) Show that if S is a first-order alphabet and $\phi_1, \ldots, \phi_n, \phi'_1, \ldots, \phi'_m \in L^S$ then if $\phi_1 \cdots \phi_n = \phi'_1 \cdots \phi'_m$ as words in L^S then $n = m$ and $\phi_i = \phi'_i$ for $i = 1, \ldots, n$.

Solution. (a) Let P be the property which holds for an S-formula φ if and only if *for all S-formulas ψ, ψ is not a prefix of φ and φ is not a prefix of ψ*. We now proceed by induction, and first the base case(s).

If $\phi = t_1 \equiv t_2$ and ψ is a prefix of ϕ, then there exists an α such that $\phi = \psi\alpha$. Without loss of generality[3] we can take $\psi = t'_1 \equiv t'_2$ to have $t_1 \equiv t_2 = t'_1 \equiv t'_2\alpha$. This equality implies $t_1 = t'_1$ and $t_2 = t'_2\alpha$. By Lemma 4.2(a) from the text, $\alpha = \Box$ (the empty string), and hence $\psi \equiv \phi$. Thus S-formulas of the form $\phi = t_1 \equiv t_2$ have property P.

If $\phi = Rt_1 \cdots t_n$ and ψ is a prefix of ϕ, then there exists an α such that $\phi = \psi\alpha$. Since the relation symbol is the first symbol we must have $\psi = Rt'_1 \cdots t'_n$. We then have $Rt_1 \cdots t_n = Rt'_1 \cdots t'_n\alpha$, and stripping away the relation[4] we have $t_1 \cdots t_n = t'_1 \cdots t'_n\alpha$. Again by Lemma 4.2(a) from the text $\alpha = \Box$ and $\psi \equiv \phi$. Thus S-formulas of the form $\phi = Rt_1 \cdots t_n$ have property P.

Moving on to the induction step we look at $\neg\phi$ which clearly has property P inheriting from ϕ, combined with the fact that $\neg$ is not an S-formula. Next we look at S-formulas of the form $\phi * \psi$ where $* = \wedge, \vee, \rightarrow, \leftrightarrow$ where ϕ and ψ both enjoy the property P. Let χ be a prefix such that $\phi * \psi = \chi\alpha$. Since $*$ must appear on the right hand side it can either appear in χ or α. In the first case we have $\phi * \psi = \beta * \gamma\alpha$ and hence $\psi = \gamma\alpha$ forcing $\alpha = \Box$ and $\psi = \gamma$. In the second case we have $\phi * \psi = \chi\beta * \gamma$ and hence $\phi = \chi\beta$ forcing $\beta = \Box$ and $\phi = \chi$. Thus all S-formulas $\phi * \psi$ do not have S-formula-prefixes.

Finally we have $\forall x\phi$ and $\exists x\phi$ which enjoy P because $\forall x$ and $\exists x$ are not S-formulas, and ϕ enjoys P.

(b)

[3] is this true?

[4] Is this allowed? If so, by what means?

Problem 3: Uniqueness decomposition

Let S be a first-order alphabet, $n \geq 1$, and $t_1, \ldots, t_n \in T^S$. If $w = t_1 \cdots t_n \in S^*$. Show by induction that for each $i < |w|$, there is a unique term $t \in T^S$ and unique $v \in S^*$ such that $w = w[1..i]tv$.

Solution.