Open Quantum Systems Assignment 1

Name: Nate Stemen (20906566)
Email: `nate@stemen.email`
Due: Mon, Feb 22, 2020 11:59 PM
Course: AMATH 876

Exercise 2.2

Show that the reduced state obtained via partial trace is a density operator, i.e., a non-negative operator satisfying $\operatorname{tr}\hat{\rho}_A = 1$.

Solution. Take $\hat{\rho}_{AB}$ to be a density operator on Hilbert space $\mathcal{H}_{AB}$. In a (tensor product) basis this looks like

$$\hat{\rho}_{AB} = \sum_{i,j,k,\ell} p_{ijk\ell} \left|a_i\right\rangle \otimes \left|b_j\right\rangle \left\langle a_k\right| \otimes \left\langle b_\ell\right|$$

In order to get a condition on it's coefficients $p_{ijk\ell}$ we'll take it's trace and force it to be 1.

$$\begin{aligned} \operatorname{tr}\hat{\rho}_{AB} &= \sum_{n,m} \left\langle a_n\right| \otimes \left\langle b_m\right| \left[\hat{\rho}_{AB}\right] \left|a_n\right\rangle \otimes \left|b_m\right\rangle \\ &= \sum_{n,m,i,j,k,\ell} p_{ijk\ell}\, \delta_{in}\, \delta_{jm}\, \delta_{kn}\, \delta_{\ell m} \\ &= \sum_{i,j} p_{ijij} = 1 \end{aligned} \tag{$*$}$$

We'll start by computing ρ_A in this basis.

$$\begin{aligned} \operatorname{tr}_B \hat{\rho}_{AB} &= \sum_n \mathbb{1} \otimes \left\langle b_n\right| \left[\hat{\rho}_{AB}\right] \mathbb{1} \otimes \left|b_n\right\rangle \\ &= \sum_{n,i,j,k,\ell} p_{ijk\ell}\, \delta_{nj}\, \delta_{\ell n} \left|a_i\right\rangle\!\left\langle a_k\right| \\ &= \sum_{n,i,k} p_{inkn} \left|a_i\right\rangle\!\left\langle a_k\right| \end{aligned}$$

Now we can ensure $\operatorname{tr}\hat{\rho}_A = 1$.

$$\begin{aligned} \operatorname{tr}\hat{\rho}_A &= \sum_\ell \left\langle a_\ell\right| \left[\hat{\rho}_A\right] \left|a_\ell\right\rangle \\ &= \sum_{\ell,n,i,k} p_{inkn}\, \delta_{\ell i}\, \delta_{k\ell} \\ &= \sum_{\ell,n} p_{\ell n \ell n} \end{aligned}$$

Hence by eq. ($*$) we can conclude $\operatorname{tr}\hat{\rho}_A = 1$.

To show $\hat{\rho}_A$ is positive semi-definite we can use the fact that the sum of the diagonal terms of a density operator are both real and positive. Real because density operators are Hermitian and positive because they are probabilities.

$$\begin{aligned} \langle \phi | \hat{\rho}_A | \phi \rangle &= \sum \overline{\phi}_n \langle a_n | \, p_{ijkj} \, | a_i \rangle\langle a_k | \, \phi_n \, | a_n \rangle \\ &= \sum |\phi_n|^2 p_{ijkj} \, \delta_{ni} \, \delta_{kn} \\ &= \sum_{ij} \underbrace{|\phi_i|^2}_{\in[0,1]} \underbrace{p_{ijij}}_{\in\mathbb{R}} \\ &\geq \sum_{ij} p_{ijij} = 1 \geq 0 \end{aligned}$$

Hence we've shown the partial trace $\mathrm{tr}_B : \mathrm{L}(H_A \otimes H_B) \to \mathrm{L}(H_A)$ preserves density operators.

Exercise 2.3

Prove that these three pure-state conditions are equivalent.

Solution. First we'll show $\hat{\rho}^2 = \hat{\rho} \implies \operatorname{tr}\hat{\rho}^2 = 1$.

$$\operatorname{tr}\hat{\rho}^2 = \operatorname{tr}\hat{\rho} = 1$$

Next we have $\operatorname{tr}\hat{\rho}^2 = 1 \implies \hat{\rho} = |\psi\rangle\langle\psi|$. Let's first calculate $\hat{\rho}^2$ in an othormal basis.

$$\begin{aligned}
\hat{\rho}^2 &= \left(\sum_{i=1}^{n} p_i |i\rangle\langle i|\right)^2 = \sum_{i=1}^{n} p_i |i\rangle\langle i| \sum_{j=1}^{n} p_j |j\rangle\langle j| \\
&= \sum_{i,j=1}^{n} p_i p_j |i\rangle \langle i|j\rangle \langle j| && \langle i|j\rangle = \delta_{ij} \\
&= \sum_{i=1}^{n} p_i^2 |i\rangle\langle i|
\end{aligned}$$

Now taking the trace of $\hat{\rho}^2$ we get a condition on the p_i's.

$$\begin{aligned}
\operatorname{tr}\hat{\rho}^2 &= \sum_{k=1}^{n} \langle k| \hat{\rho}^2 |k\rangle \\
&= \sum_{k,i=1}^{n} \langle k| \left(p_i^2 |i\rangle\langle i|\right) |k\rangle \\
&= \sum_{i=1}^{n} p_i^2 = 1
\end{aligned}$$

So $\sum p_i = 1 = \sum p_i^2$. By basic properties of real numbers, we know $p_i^2 \leq p_i$ when $p_i \in [0, 1]$ and equality only holding when $p_i \in \{0, 1\}$. Using this fact we can write

$$\sum_{i=1}^{n} p_i^2 \leq \sum_{i=1}^{n} p_i$$

where again equality only holds if $p_i \in \{0, 1\}$ for all i. The only way this can be true is if one of the p_i's is 1 and all of the rest are 0. In that case our summation collapses to one term, and we are left with $\hat{\rho} = |i\rangle\langle i|$ as desired.

Lastly we have $\hat{\rho} = |\psi\rangle\langle\psi| \implies \hat{\rho}^2 = \hat{\rho}$.

$$\hat{\rho}^2 = (|\psi\rangle\langle\psi|)(|\psi\rangle\langle\psi|) = |\psi\rangle \underbrace{\langle\psi|\psi\rangle}_{1} \langle\psi| = |\psi\rangle\langle\psi| = \hat{\rho}$$

Exercise 2.5

Prove the existence of the Schmidt decomposition.

Solution. Let $\{|a_i\rangle\}$ be an orthonormal basis for H_A and $\{|b_i\rangle\}$ be an orthonormal basis for H_B. We then know $|a_n\rangle \otimes |b_m\rangle$ forms a basis for $H_{AB} = H_A \otimes H_B$, and hence we can expand any vector $|\psi\rangle \in H_{AB}$ as

$$|\psi\rangle = \sum_{ij} \psi_{ij} |a_i\rangle \otimes |b_j\rangle .$$

We can think of the coefficients ψ_{ij} as a matrix using the association $|a_i\rangle \otimes |b_j\rangle \cong |a_i\rangle\langle b_j|$.[1] This operator is then taking an element of H_B to an element of H_A. That is we can think of the *vector* $|\psi\rangle$ as a linear map $|\psi\rangle_{\text{op}} : H_B \to H_A$.

With this picture in place we can apply the singular decomposition to write

$$|\psi\rangle_{\text{op}} = \sum_{n=1}^{r} \lambda_n |a_n\rangle\langle b_n|$$

where r is the rank of $|\psi\rangle_{\text{op}}$ the *operator*. Now we can map back into $|\psi\rangle$ again using $|a_n\rangle\langle b_m| \cong |a_n\rangle \otimes |b_m\rangle$ and see we have

$$|\psi\rangle = \sum_{n=1}^{r} \lambda_n |a_n\rangle \otimes |b_n\rangle .$$

[1] Using the underlying isomorphism of $U^* \otimes V$ and the space $\text{Hom}(U, V)$ of linear maps from U to V.

Exercise 3.1

Prove the two properties given by Eqns. 3.1.

Solution. First we'll show $\sum_\nu E_\nu = \mathbb{1}$.

$$\begin{aligned}\sum_\nu E_\nu &= \sum_\nu \text{tr}_B \left(\Pi_\nu \cdot \mathbb{1}_A \otimes \hat{\rho}_B\right) \\ &= \text{tr}_B \left(\sum_\nu \Pi_\nu \cdot \mathbb{1}_A \otimes \hat{\rho}_B\right) \\ &= \text{tr}_B \left(\left[\sum_\nu \Pi_\nu\right] \cdot \mathbb{1}_A \otimes \hat{\rho}_B\right) \\ &= \text{tr}_B(\mathbb{1}_A \otimes \hat{\rho}_B) \\ &= \mathbb{1}_A\end{aligned}$$

$\sum_\nu \Pi_\nu = \mathbb{1}_{A\otimes B}$

Now we'll show $E_\nu \geq 0$, but first we need some setup. Since Π_ν is a projector, by the spectral theorem it can be written as

$$\Pi_\nu = \sum_{ij} \lambda_{ij} \left|a_i b_j\right\rangle\!\left\langle a_i b_j\right|$$

where λ_{ij} are real and non-negative. We also have $\hat{\rho}_B = \sum p_n \left|b_n\right\rangle\!\left\langle b_n\right|$ and $\mathbb{1}_A = \sum \left|a_n\right\rangle\!\left\langle a_n\right|$.

$$\begin{aligned}\Pi_\nu \cdot \mathbb{1}_A \otimes \hat{\rho}_B &= \sum \lambda_{ij} p_n \left|a_i b_j\right\rangle \left\langle a_i b_j \middle| a_\ell b_n\right\rangle \left\langle a_\ell b_n\right| \\ &= \sum \lambda_{ij} p_n \, \delta_{i\ell}\, \delta_{jn} \left|a_i b_j\right\rangle\!\left\langle a_\ell b_n\right| \\ &= \sum_{\ell n} \lambda_{\ell n} p_n \left|a_\ell b_n\right\rangle\!\left\langle a_\ell b_n\right|\end{aligned}$$

Now we can take the partial trace over B.

$$\begin{aligned}\text{tr}_B \left(\Pi_\nu \cdot \mathbb{1}_A \otimes \hat{\rho}_B\right) &= \sum \mathbb{1}_A \otimes \left\langle b_i\right| \lambda_{\ell n} p_n \left|a_\ell b_n\right\rangle\!\left\langle a_\ell b_n\right| \mathbb{1} \otimes \left|b_i\right\rangle \\ &= \sum \lambda_{\ell n} p_n \left|a_\ell\right\rangle\!\left\langle a_\ell\right| \; \delta_{in}\, \delta_{ni} \\ &= \sum_{\ell n} \lambda_{\ell n} p_n \left|a_\ell\right\rangle\!\left\langle a_\ell\right|\end{aligned}$$

Now finally we can can take the expectation values to show E_ν is positive semi-definite.

I'm a little iffy if this actually works. I feel like there should be a more elegant way to show E_ν is positive semi-definite, but I can't come up with anything.

Exercise 3.2

Apply Naimark's theorem to identify a PVM[a] in an extended Hilbert space that generates the trine.

[a]Projection-Valued Measure

Solution. We'd like to find a set of operators $\{F_i\}$ such that $F_i F_j = F_i \delta_{ij}$ and $\sum_i F_i = \mathbb{1}$ that are built from the operators E_ν. First recall the trine states:

$$|\chi_0\rangle = |0\rangle \qquad |\chi_1\rangle = \frac{1}{2}|0\rangle - \frac{\sqrt{3}}{2}|1\rangle \qquad |\chi_2\rangle = \frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle$$

Now we can calculate the POVM[2]s as follows.

$$\begin{aligned}
E_0 &= \frac{2}{3}|\chi_0\rangle\langle\chi_0| = \frac{2}{3}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \\
E_1 &= \frac{1}{6}\Big(|0\rangle\langle 0| - \sqrt{3}\,|0\rangle\langle 1| - \sqrt{3}\,|1\rangle\langle 0| + 3\,|1\rangle\langle 1|\Big) = \frac{1}{6}\begin{bmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & 3 \end{bmatrix} \\
E_2 &= \frac{1}{6}\Big(|0\rangle\langle 0| + \sqrt{3}\,|0\rangle\langle 1| + \sqrt{3}\,|1\rangle\langle 0| + 3\,|1\rangle\langle 1|\Big) = \frac{1}{6}\begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & 3 \end{bmatrix}
\end{aligned}$$

Now we'll look for a U satisfying $U^\dagger(\mathbb{1} \otimes E_\nu)U$. Indeed we can define (by the positiveness of E_ν)

$$U = \sqrt{E_0} \otimes \mathbf{e}_0 + \sqrt{E_1} \otimes \mathbf{e}_1 + \sqrt{E_2} \otimes \mathbf{e}_2.$$

Where $\mathbf{e}_i$ are the standard basis elements of $\mathbb{C}^3$. This is indeed an isometry ($U^\dagger U = \mathbb{1}$) as

$$\begin{aligned}
U^\dagger U &= \Big(\sqrt{E_0} \otimes \mathbf{e}_0^\dagger + \sqrt{E_1} \otimes \mathbf{e}_1^\dagger + \sqrt{E_2} \otimes \mathbf{e}_2^\dagger\Big)\Big(\sqrt{E_0} \otimes \mathbf{e}_0 + \sqrt{E_1} \otimes \mathbf{e}_1 + \sqrt{E_2} \otimes \mathbf{e}_2\Big) \\
&= E_0 \otimes \underbrace{\mathbf{e}_0^\dagger \mathbf{e}_0}_{1} + E_1 \otimes \mathbf{e}_1^\dagger \mathbf{e}_1 + E_2 \otimes \mathbf{e}_2^\dagger \mathbf{e}_2 \\
&= E_0 + E_1 + E_2 = \mathbb{1}
\end{aligned}$$

Thus we take $F_i = U^\dagger(\mathbb{1} \otimes E_i)U$.

[2]Positive Operator-Valued Measure

Exercise 3.3

(a) Verify that the map E defined in terms of projectors onto coherent states in the example above satisfies the postulates of a POVM.
(b) What is the operational interpretation of $\Pr(X) = \operatorname{tr}(E(X)\rho)$ for this POVM, noting that $\alpha = (\langle q \rangle, \langle p \rangle)$ denotes the expectations of the position q and momentum p operatores in the associated coherent state, and that $\Omega = \mathbb{R}^2$ means we are measuring the position and momentum of some particle?

Solution. (a) To show $E(X)$ defines a valid POVM we need to show

1. $E(X) \geq 0$
2. $E(\mathbb{R}^2) = 1$
3. $E(\bigcup_i X_i) = \sum_i E(X_i)$

1

$$\langle \psi | E(X) | \psi \rangle = \frac{1}{\pi} \int_X \mathrm{d}^2\alpha \, \langle \psi | \alpha \rangle \langle \alpha | \psi \rangle = \frac{1}{\pi} \int_X \mathrm{d}^2\alpha \, \underbrace{|\langle \alpha | \psi \rangle|^2}_{\geq 0} \geq 0$$

This applys for all vectors $|\psi\rangle$, so we conclude $E(X)$ is positive semi-definite.

2

This point is satisfied by the resolution of the identity given in the example prior to the question:

$$E(\mathbb{R}^2) = \frac{1}{\pi} \int_{\mathbb{R}^2} \mathrm{d}^2\alpha \, |\alpha\rangle\langle\alpha| = \mathbb{1}$$

3

Because the X_i are disjoint, the additivity of the Lebesgue integral allow us to split the integral into pieces as follows.

$$E\left(\bigcup_i X_i\right) = \frac{1}{\pi} \int_{\bigcup_i X_i} \mathrm{d}^2\alpha \, |\alpha\rangle\langle\alpha| = \sum_i \frac{1}{\pi} \int_{X_i} \mathrm{d}^2\alpha \, |\alpha\rangle\langle\alpha| = \sum_i E(X_i)$$

The convergence of the (possibly) infinite sum is guaranteed by first using the above manipulation on the first n sets X_i and then taking the limit.

(b) The operational interpretation of $\Pr(X) = \operatorname{tr}(E(X)\rho)$ is that this is the probability of finding the particle in and maybe around the state space region X.