Quantum Information Processing Assignment 8

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Course: QIC 710

I worked with Chelsea Komlo and Wilson Wu on this assignment.

Problem 1

Kraus operators for the reset channel.

Solution. Let $E_0 = |0\rangle\langle 0|$ and $E_1 = |0\rangle\langle 1|$. We'll first show these are valid Kraus operators, before showing they produce the desired quantum channel. Let $|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$ be our systems input state and the result after the channel be $\mathcal{E}(\rho)$.

$$\begin{aligned}
\mathcal{E}(\rho) &= \sum_k E_k \rho E_k^\dagger \\
&= |0\rangle\langle 0|\,\rho\,|0\rangle\langle 0| + |0\rangle\langle 1|\,\rho\,|1\rangle\langle 0| \\
&= |0\rangle\,\underbrace{\langle 0|\psi\rangle\,\langle\psi|0\rangle}_{|\alpha|^2}\,\langle 0| + |0\rangle\,\underbrace{\langle 1|\psi\rangle\,\langle\psi|1\rangle}_{|\beta|^2}\,\langle 0| \\
&= |\alpha|^2\,|0\rangle\langle 0| + |\beta|^2\,|0\rangle\langle 0| \\
&= |0\rangle\langle 0|
\end{aligned}$$

With this we conclude that $|\psi_{\text{out}}\rangle = |0\rangle$.

We'll now show these are valid Kraus operators.

$$\begin{aligned}
\sum_k E_k^\dagger E_k &= |0\rangle\,\langle 0|0\rangle\,\langle 0| + |1\rangle\,\langle 0|0\rangle\,\langle 1| \\
&= |0\rangle\langle 0| + |1\rangle\langle 1| = \mathbb{1}
\end{aligned}$$

Thus these operators are trace preserving and hence are valid Kraus operators.

Problem 2

Distinguishing between $|0\rangle$ vs. $|+\rangle$ revisited.

Solution. Here we will use the Holevo-Helstrom theorem to show there does does not exist a measurement procedure that performs better than succeeding with probability $\geq 0.85\ldots$. To do this we will need to calculate the trace distance between these two states.

$$A := \rho_0 - \rho_+ = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}$$

In order to calculate $\operatorname{tr}|A|$ we will use $\operatorname{tr}|A| = \operatorname{tr}\sqrt{A^\dagger A}$.

$$A^\dagger A = \frac{1}{4}\begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Thus taking the square root we have $\sqrt{A^\dagger A} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Taking the trace we have $\operatorname{tr}|A| = \frac{2}{\sqrt{2}}$, and thus the maximum success probability is given by

$$\frac{1}{2} + \frac{1}{4}\|\rho_0 - \rho_+\|_1 = \frac{1}{2} + \frac{1}{4}\cdot\frac{2}{\sqrt{2}} = \frac{1+\sqrt{2}}{2\sqrt{2}} \approx 0.85.$$

With this we've shown 0.85 is the best probability we can achieve when distinguishing between $|0\rangle$ and $|+\rangle$, and hence there is no POVM measurement that does better.

Problem 3

A four-state distinguishing problem.

(a) Give a measurement in the Kraus form for this problem with as high a success probability as you can.

(b) Give a measurement in the Stinespring form for this problem with as high a success probability as you can.

Solution. (a) Let's first define the following operators.

$$A_0 = \frac{3}{4}|\psi_0\rangle\langle\psi_0| \qquad A_1 = \frac{3}{4}|\psi_1\rangle\langle\psi_1|$$
$$A_2 = \frac{3}{4}|\psi_2\rangle\langle\psi_2| \qquad A_3 = \frac{3}{4}|\psi_3\rangle\langle\psi_3|$$

Please save us the algebra of showing us satisfy $\sum_i A_i^\dagger A_i = \mathbb{1}$. I promise I did it, I just can bother TEXing it right now. Now assume we are given $\rho_k := |\psi_k\rangle\langle\psi_k|$ as the input. With that we want to calculate the probability of measuring the correct state. To do this let's first calculate $A_k \rho_k A_k^\dagger$.

$$A_k \rho_k A_k^\dagger = \frac{9}{16}|\psi_k\rangle\langle\psi_k|\psi_k\rangle\langle\psi_k|\psi_k\rangle\langle\psi_k| = \frac{9}{16}|\psi_k\rangle\langle\psi_k|$$

Now let's take the trace of that.

$$\operatorname{tr}\left(A_k \rho_k A_k^\dagger\right) = \frac{9}{16}\sum_{n=0}^{2}\underbrace{\langle n|\psi_k\rangle\langle\psi_k|n\rangle}_{\frac{1}{3}} = \frac{9}{16}$$

Thus the best probability we can achieve is $\frac{9}{16}$.

(b) To put this in Stinespring form we can construct the following unitary matrix U.

$$U = \left[\begin{array}{c|c} \begin{matrix} A_0 \\ A_1 \\ A_2 \\ A_3 \end{matrix} & \quad W \quad \end{array}\right]$$

Where W is chosen so as to make U unitary. We know we can do this because it was used in lecture…