The Gottesman-Knill Theorem

What is it and what does it mean?

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QIC 710 Final Project

The Gottesman-Knill Theorem

Theorem ([Gottesman, 1998])

*A quantum circuit using only the following elements can be efficiently **simulated** on a classical computer:*

1. *Qubits prepared in computational basis states*
2. *Quantum gates from the **Clifford group***
3. *Measurements in the computational basis*

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Given an input x to our quantum computer, compute $p(x)$.

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Weak Simulation

Given an input x, compute a sample from $p(x)$.

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Strong Simulation

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Weak Simulation

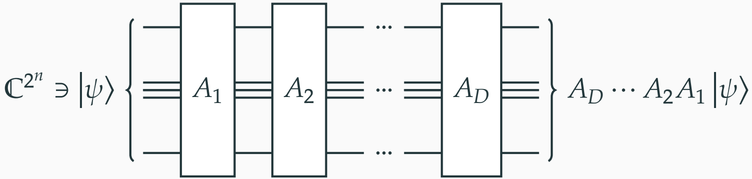
Given an input x, compute a sample from $p(x)$.

Gottesman-Knill theorem deals with weak simulation.

Strong simulation of quantum computers shown to be #**P**-hard [Nest, 2010].

How can we (naïvely) simulate a quantum computer?

Suppose we have n qubits and we want to run them through D gates.



Final state contains $D-1$ matrix multiplications, each costing $O\left(2^{3n}\right)$[1], so total cost is $O\left(D2^{3n}\right)$.

Simulating Grover's algorithm on 40 qubits took nearly a full day! [Viamontes et al., 2004]

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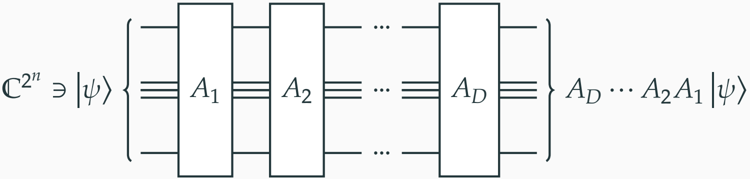

What if we restrict the gates A_i?

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Stabilizer Formalism

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- $\left|\psi\right\rangle$ is the *unique* state stabilized by both of these operators.
- This hints at the possibility of describing some states not as vectors in $\mathbb{C}^{2^n}$, but of operators.

Pauli Group

Let X, Y, Z denote the standard single-qubit Pauli operators:

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

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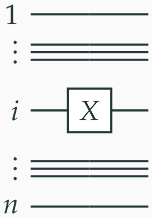
Take X_i, Y_i, Z_i to denote X, Y and Z acting on the i-th qubit, and with the identity everywhere else.

$$X_i := \mathbb{1} \otimes \cdots \otimes \overbrace{X}^{i\text{th operator}} \otimes \cdots \otimes \mathbb{1}$$



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Take X_i, Y_i, Z_i to denote X, Y and Z acting on the i-th qubit, and with the identity everywhere else.

$$P_n := \{\pm\mathbb{1}, \pm\mathrm{i}\mathbb{1}, \pm A_i, \pm\mathrm{i}A_i : A_i \in \{\mathbf{1}, X_i, Y_i, Z_i\}\} \equiv \langle X_i, Z_i \rangle$$

- P_n forms a group under matrix multiplication.
- Every pair of elements either commute or anti-commute.
- $|P_n| = 4 \cdot 4^n$

Stabilizer States

Let S be a subgroup of P_n. Define the *vector space* V_S as the states stabilized by everything in S.

$$V_S := \left\{ \left|\psi\right\rangle \in \mathbb{C}^{2^n} : g\left|\psi\right\rangle = \left|\psi\right\rangle, \forall g \in S \right\}$$

Example

Take P_3 and subgroup $S = \{\mathbb{1}, Z_1Z_2, Z_2Z_3, Z_1Z_3\}$. Note that $|000\rangle, |001\rangle, |110\rangle, |111\rangle$ are stabilized by Z_1Z_2, and $|000\rangle, |100\rangle, |011\rangle, |111\rangle$ are stabilized by Z_2Z_3. These, together with the fact that $Z_1Z_3 = (Z_1Z_2)(Z_2Z_3)$ tell us that $V_S = \{|000\rangle, |111\rangle\}$. In this case we can write $S = \langle Z_1Z_2, Z_2Z_3\rangle$.

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S and V_S uniquely determine each other!

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- S must be Abelian
- $-\mathbb{1} \notin S$
- $|S| = 2^{n-k}$ for some $k < n$

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Corollary

If S is generated by $g_1, \dots, g_n$, then $USU^\dagger$ is generated by $Ug_1U^\dagger, \dots, Ug_nU^\dagger$.

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The *Clifford Group* is defined to be the set of operators that leave Pauli operators invariant under conjugation.

$$C_n := \left\{ V \in \mathbf{U}(2^n) : VP_nV^\dagger = P_n \right\}$$

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$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad P = \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{i} \end{pmatrix}$$

CNOT

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4. Keeping track of the generators of a stabilizer S provide a succinct way to understand how S is changing ($\log |S|$ generators)
5. Found that elements of the Clifford group can efficiently build elements that conjugate the Pauli group back to the Pauli group

Back to the Theorem

Theorem ([Gottesman, 1998])

A quantum circuit using only the following elements can be efficiently simulated on a classical computer:

1. *Qubits prepared in computational basis states*
2. *Quantum gates from the Clifford group*
3. *Measurements in the computational basis*

- Take $\left|\psi\right\rangle = |0\rangle^{\otimes n}$. Now we can say $S = \langle Z_1, \dots, Z_n \rangle$.
- Under some action $U \in C_n$ state will evolve to $U\left|\psi\right\rangle = UgU^{\dagger}U\left|\psi\right\rangle$ for $g \in S$
- Switch over to describing the change in generators of S
- Need to compute $UZ_1U^{\dagger}, \dots, UZ_nU^{\dagger}$

Back to the Theorem

Recap

We have $|\psi\rangle = |0\rangle^{\otimes n}$, $S = \langle Z_1, \dots, Z_n \rangle$, and $U \in C_n$. We know that $U|\psi\rangle = g'U|\psi\rangle$, so in order to figure out where it evolves to, need to compute the new generators of the space UV_S which are $UZ_1U^\dagger, \dots, UZ_nU^\dagger$.

- Only takes $2n+1$ bits to keep track of a generator: 2 for the n Pauli generators, and 1 for the phase of ± 1
- Specifying $|\psi\rangle$ requires all n generators, so $n(2n+1)$ bits
- Updating the generators only takes $O(n)$
- Total cost $O(n^2)$

X_1	1
X_2	0
X_3	1
Z_1	0
Z_2	0
Z_3	1
± 1	0

Table 1: Encoding $X_1X_3Z_3$

What does this mean?

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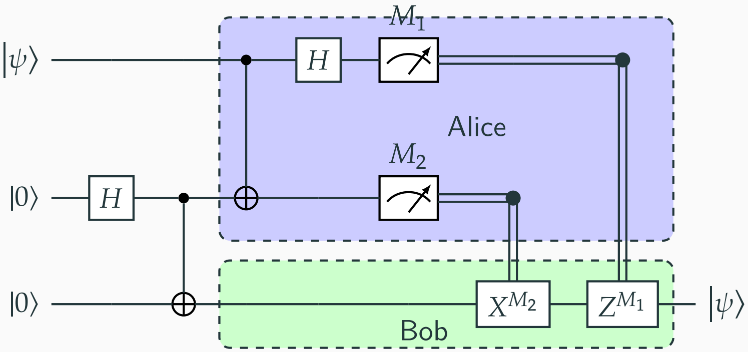
Quantum teleportation

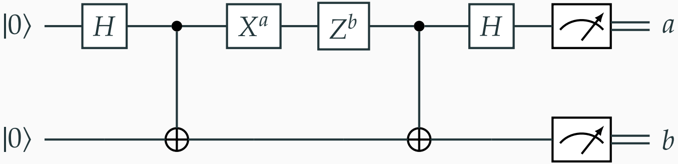


What does this mean?

What makes quantum computers powerful?

Not *just* entanglement!

Superdense coding



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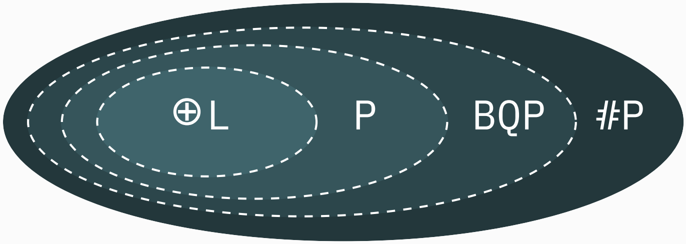
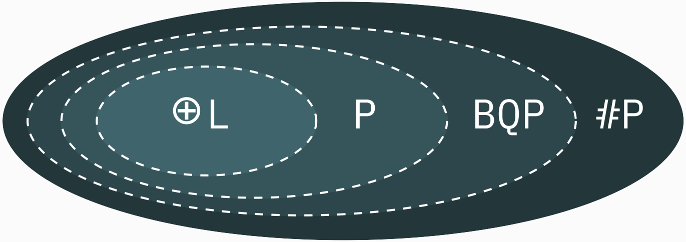


Where do we go from here?

- Clifford gates aren't enough for universal quantum computation
- Clifford group shown to be ⊕**L**-complete [Aaronson and Gottesman, 2004]
- Adding *any* 1 or 2-qubit gate[2] will turn the Cliffords into a universal set [Shi, 2002]



[2]That doesn't map computational basis states to computational basis states

Conclusion

Assuming

$\mathbf{BQP} \neq \mathbf{P} \neq \oplus\mathbf{L}$

Strong simulation of quantum computers is *really* hard.

Clifford group is only capable of solving relatively easy problems (both from classical and quantum POV).

Quantum entanglement is not the only contributing factor to the power of quantum computers!

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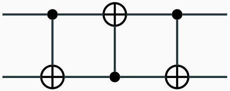
☺**Thank you!**☺

Questions?

Example

Alice's Broken Quantum Computer ☹

Alice's quantum computer is working too well. Instead of performing single controlled-NOT gates, it does three at a time. What is it actually doing?



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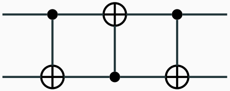

Because X_1, X_2, Z_1, Z_2 generate the Pauli group we can follow what happens to them under the evolution of this circuit.

$$X_1 = X \otimes \mathbb{1} \xrightarrow{\text{CNOT 1}} \text{CNOT} \cdot (X \otimes \mathbb{1}) \cdot \text{CNOT}^{\dagger} = X \otimes X$$

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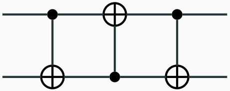

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$$X_1 = X \otimes \mathbb{1} \xrightarrow{\text{CNOT 1}} X \otimes X \xrightarrow{\text{CNOT 2}} \mathbb{1} \otimes X \xrightarrow{\text{CNOT 3}} \mathbb{1} \otimes X = X_2$$

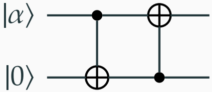
$$Z_1 = Z \otimes \mathbb{1} \xrightarrow{\text{CNOT 1}} Z \otimes \mathbb{1} \xrightarrow{\text{CNOT 2}} Z \otimes Z \xrightarrow{\text{CNOT 3}} \mathbb{1} \otimes Z = Z_2$$

Further, we can show $X_1 \longleftrightarrow X_2$ and $Z_1 \longleftrightarrow Z_2$. This is exactly a swap operation!

Example, continued

Alice's less Broken Quantum Computer ☺

By dint of no little hard work, Alice has partially fixed her quantum computer. Now it only does 2 CNOTs at a time. Unfortunately, she can only get this improvement if she puts a $|0\rangle$ as the second input qubit. What does it do now?



In this case we see the initial state $|\psi_0\rangle = |\alpha\rangle \otimes |0\rangle$ is stabilized by Z_2.

State will always be a +1 eigenvector of Z_2, so it follows that $(Z \otimes \mathbb{1})(Z \otimes Z) = \mathbb{1} \otimes Z$.

Circuit still performs a swap!

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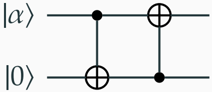

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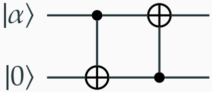

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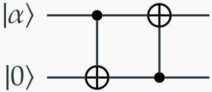

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