Advanced Quantum Theory Homework 4

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Exercise 24

Let us consider how large we can make the domain $D_f \subset \mathcal{H}$ of the operator $\hat{f}$ defined in Eq. 3.98. Concretely, how fast do the coefficients ψ_n of $|\psi\rangle$ have to decay, as $n \to \infty$. For this exercise, assume that the coefficients $|\psi_n|$ scale as n^s for $n \to \infty$. What then are the allowed values for s? Hint: The so-called Dirichlet series representation of the Riemann zeta function $\zeta(r) := \sum_{n=1}^{\infty} 1/n^r$ diverges for $r \leq 1$ butconverges for $r > 1$.

Solution. Lets first expand $|\varphi\rangle = \hat{f}\,|\psi\rangle$ so we can see how it scales with n.

$$\begin{aligned} |\varphi\rangle = \hat{f}\,|\psi\rangle &= \begin{bmatrix} 1^2 & & & \\ & 2^2 & & \\ & & 3^2 & \\ & & & \ddots \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \vdots \end{bmatrix} \\ &= \begin{bmatrix} 1^2\psi_1 \\ 2^2\psi_2 \\ 3^2\psi_3 \\ \vdots \end{bmatrix} \end{aligned}$$

It's then clear that $\varphi_n = n^2\psi_n$. Now we need $\sum_n \overline{\varphi}_n \varphi = \sum_n n^4 \overline{\psi}_n \psi_n$ to converge, and if ψ_n scales as n^{-s} (sorry changing the question a little bit), then s must be greater than 2.5. Let $s = 2.5 + \frac{\varepsilon}{2}$ where $\varepsilon \in \mathbb{R}_+$[1]. Now lets expand the norm.

$$\begin{aligned} \|\varphi\|^2 &= \sum_n n^4 \left(\frac{1}{n^{2.5+\frac{\varepsilon}{2}}}\right)^2 \\ &= \sum_n n^4 \frac{1}{n^{5+\varepsilon}} \\ &= \sum_n \frac{1}{n^{1+\varepsilon}} \end{aligned}$$

Now because we've chosen $\varepsilon > 0$, we know this will converge and hence the state can be normalized. So $|\psi_n|$ must scale like $\frac{1}{n^s}$ for $s > 2.5$.

This means the domain D_f can act on all vectors that when $n \to \infty$ they go to zero faster than $\frac{1}{n^{2.5}}$.

[1] Using the convention here that $\mathbb{R}_+ = \{x \in \mathbb{R} \mid x > 0\}$.

Exercise 25

By comparing Eqs. 3.108 and Eq. 3.109, find out how to express the wave function of $|\psi\rangle$ in the b basis in terms of its wave function in the c basis, using the matrix of $\hat{U}$. Also express the representation of $\hat{f}$ as a matrix in the b basis in terms of its representation as a matrix in the c basis, using the matrix representation of $\hat{U}$. Finally write Eq. 3.109 in terms of a row vector, some matrices and a column vector.

Solution. First, a basis change of $|\psi\rangle$ in the b basis to the c basis.

$$\begin{aligned} |\psi\rangle &= \sum_n \psi_n |b_n\rangle \\ &= \sum_n \psi_n \mathbb{1} |b_n\rangle \\ &= \sum_{n,m} \psi_n (|c_m\rangle\langle c_m|) |b_n\rangle \\ &= \sum_{n,m} \psi_n \langle c_m|b_n\rangle |c_m\rangle \\ &= \sum_{n,m} \psi_n U_{mn} |c_m\rangle \\ &= \sum_m \tilde{\psi}_m |c_m\rangle \end{aligned}$$

Next, we do the same for an operator $\hat{f}$.

$$\begin{aligned} \hat{f} &= \sum_{i,j} |b_i\rangle f_{ij} \langle b_j| \\ &= \sum_{i,j} \mathbb{1} |b_i\rangle f_{ij} \langle b_j| \mathbb{1} \\ &= \sum_{i,j,k,l} (|c_k\rangle\langle c_k|) |b_i\rangle f_{ij} \langle b_j| (|c_l\rangle\langle c_l|) \\ &= \sum_{i,j,k,l} |c_k\rangle \left[\langle c_k|b_i\rangle f_{ij} \langle b_j|c_l\rangle\right] \langle c_l| \\ &= \sum_{i,j,k,l} |c_k\rangle \left[U_{ki} f_{ij} U_{jl}\right] \langle c_l| \\ &= \sum_{i,j} |c_i\rangle \tilde{f}_{ij} \langle c_j| \end{aligned}$$

Lastly, the matrix/vector portion of this question. Here we write out the term being summed over, ommitting the summation for brevity.

$$\underbrace{\langle\psi|}_{\psi^\dagger} \underbrace{|c_r\rangle \overbrace{\langle c_r|b_n\rangle}^{\overline{U_{nr}}} \langle b_n|}_{\hat{U}^\dagger} \underbrace{|c_s\rangle \overbrace{\langle c_s|\hat{f}|c_u\rangle}^{f_{su}} \langle c_u|}_{\hat{f}} \underbrace{|b_m\rangle \overbrace{\langle b_m|c_v\rangle}^{U_m v} \langle c_v|}_{\hat{U}} \underbrace{|\psi\rangle}_{\psi}$$

Which can be written more consicely as $\psi^\dagger \, \hat{U}^\dagger \, \hat{f} \, \hat{U} \, \psi$. This can be recognized as $\hat{f}$ in the c basis because $\hat{U} |b_n\rangle \to |c_n\rangle$.

Exercise 26

Assume that the basis vectors $|b_n\rangle$ are orthonormal. Show that if the new vectors $|c_n\rangle$ are to be an orthonormal basis too then $\hat{U}$ has to obey $\hat{U}^\dagger = U^{-1}$. This means that $\hat{U}$ is what is called a unitary operator.

Solution. Let's see what happens when we take the inner product of two c_i vectors and force it to be $\delta_{n,m}$.

$$\langle c_n | c_m \rangle = \langle b_n | \hat{U}^\dagger \hat{U} | b_m \rangle$$

Given $\langle b_n | b_m \rangle = \delta_{n,m}$, then we can see if $\hat{U}^\dagger \hat{U} = \mathbb{1}$, then $\langle c_n | c_m \rangle$ will also be orthonormal as desired. If we multiply on the right with $\hat{U}^{-1}$ we obtain $\hat{U}^\dagger = \hat{U}^{-1}$ and hence $\hat{U}$ is unitary.

Exercise 27

Assume that $b, b^\dagger$ are linear maps on a Hilbert space and are obeying $\left[b, b^\dagger\right] = \mathbb{1}$,where $\mathbb{1}$ is the identity map. Assume that there is a normalized vector, which we denote by $|0\rangle$, which obeys $b\,|0\rangle = 0$. Show that the vector $|z\rangle$ which is defined through $|z\rangle = \mathrm{e}^{zb^\dagger}\,|0\rangle$ is an eigenvector of b if z is any complex number. These vectors are related to so-called coherent states which are of practical importance, for example,in quantum optics (light is often found in a similar quantum state). These states are also of importance regarding the problem of "decoherence" in quantum measurement theory, as we will discuss later.

Solution. It'll be helpful if we first write down how $|z\rangle$ is defined so we recognize it later on.

$$|z\rangle = \left[\sum_n \frac{\left(zb^\dagger\right)^n}{n!}\right]|0\rangle = \sum_n \frac{z^n}{n!}\left(b^\dagger\right)^n|0\rangle$$

Right so now let's apply b to this.

$$\begin{aligned} b\,|z\rangle &= b\,\mathrm{e}^{zb^\dagger}\,|0\rangle = b\sum_n \frac{z^n}{n!}\left(b^\dagger\right)^n|0\rangle \\ &= \sum_n \frac{z^n}{n!} b\left(b^\dagger\right)^n|0\rangle \end{aligned} \tag{1}$$

At this point we can't really do anything unless we know how to move that b in front of the $b^\dagger$ term. To this end let's use the fact that we know the commutation relation for b and it's Hermitian conjugate to calculate the commutator of b with $\left(b^\dagger\right)^n$. Here we will use the fact that $bb^\dagger = \mathbb{1} + b^\dagger b$.

$$\begin{aligned} \left[b, b^{\dagger n}\right] &= bb^{\dagger n} - b^{\dagger n}b \\ &= \left(bb^\dagger\right)b^{\dagger n-1} - b^{\dagger n}b \\ &= \left(\mathbb{1} + b^\dagger b\right)b^{\dagger n-1} - b^{\dagger n}b \\ &= b^{\dagger n-1} + b^\dagger b b^{\dagger n-1} - b^{\dagger n}b \\ &= 2b^{\dagger n-1} + b^{\dagger 2}bb^{\dagger n-2} - b^{\dagger n}b \end{aligned}$$

Here we notice a pattern emerging. Every substitution we make, we get an additional $b^{\dagger n-1}$ term, and decrease the powers of $b^\dagger$ acting on the right hand side. With this we posit $\left[b, b^{\dagger n}\right] = nb^{\dagger n-1}$ and can be verified for all n using a proof by induction. We can rewrite the commutation relation as $bb^{\dagger n} = nb^{\dagger n-1} + b^{\dagger n}b$ and you may say it doesn't seem like we're getting anywhere, and I felt that way too when working through this problem, but let's plug it into **??** and see what happens.

$$\begin{aligned} b\,|z\rangle &= \sum_n \frac{z^n}{n!} b\left(b^\dagger\right)^n|0\rangle \\ &= \sum_n \frac{z^n}{n!}\left(nb^{\dagger n-1} + b^{\dagger n}b\right)|0\rangle \end{aligned}$$

Using the given fact that $b\,|0\rangle = 0$, we can drop the second term. Let's look at this now, making sure to carefully manipulate our summation boundaries.

$$\begin{aligned}
b\,|z\rangle &= \sum_{n=0}^{\infty} \frac{z^n}{n!} n b^{\dagger n-1}\,|0\rangle \\
&= \sum_{n=1}^{\infty} \frac{z^n}{n!} n b^{\dagger n-1}\,|0\rangle && (n = 0 \text{ term is } 0) \\
&= z \sum_{n=1}^{\infty} \frac{z^{n-1}}{(n-1)!} b^{\dagger n-1}\,|0\rangle \\
&= z \sum_{n=0}^{\infty} \frac{z^n}{n!} b^{\dagger n}\,|0\rangle \\
&= z\,\mathrm{e}^{z b^\dagger}\,|0\rangle = z\,|z\rangle
\end{aligned}$$

Thus we've now shown $b\,|z\rangle = z\,|z\rangle$, so $|z\rangle$ is an eigenvector of b with eigenvalue z.

Exercise 1

Are there any values of β for which the commutation relation$[\hat{x}, \hat{p}] = \mathrm{i}\hbar\left(1 + \beta\hat{p}^2\right)$ may possess a finite-dimensional Hilbert space representation?

Solution. Previously we ruled this out as a possibility by assuming there was a finite dimensional representation, taking the trace of both sides of $[\hat{x}, \hat{p}] = \mathrm{i}\hbar$ and arriving at a contradiction. Let's try using that argument here. The trace of the left hand side will still be 0 because $\operatorname{tr}(\hat{x}\hat{p}) = \operatorname{tr}(\hat{p}\hat{x})$. That leaves $0 = \operatorname{tr}\left(\mathrm{i}\hbar\left(\mathbb{1} + \beta\hat{p}^2\right)\right)$. Using basic properties of the trace, we can expand this as follows.

$$
\begin{aligned}
0 &= \operatorname{tr}\left(\mathrm{i}\hbar\left(\mathbb{1} + \beta\hat{p}^2\right)\right) \\
&= \mathrm{i}\hbar\left(N + \beta\operatorname{tr}\left(\hat{p}^2\right)\right) && (N \text{ is dimension of Hilbert space}) \\
\beta &= -\frac{N}{\operatorname{tr}(\hat{p}^2)}
\end{aligned}
$$

Thus maybe it's possible if we have a nonlinear commutation relation. That said wouldn't $\hat{p}^2$ be different for every system, and hence β would too?

Exercise 2

Assume that $|\psi\rangle$ is an eigenvector of a self-adjoint operator $\hat{f}$ with eigenvalue λ. Show that λ is real.

Solution. The above question can simply be written as $\hat{f}\,|\psi\rangle = \lambda\,|\psi\rangle$. Let's now move some stuff around.

$$
\begin{aligned}
\hat{f}\,|\psi\rangle &= \lambda\,|\psi\rangle \\
\langle\psi|\,\hat{f}^\dagger &= \overline{\lambda}\,\langle\psi| && \text{(apply adjoint to both sides)} \\
\langle\psi|\,\hat{f}^\dagger\,|\psi\rangle &= \overline{\lambda}\,\underbrace{\langle\psi|\psi\rangle}_{1} && \text{(apply } |\psi\rangle \text{ to both sides)} \\
\langle\psi|\,\underbrace{\hat{f}\,|\psi\rangle}_{\lambda|\psi\rangle} &= \overline{\lambda} && (\hat{f} \text{ is self-adjoint}) \\
\langle\psi|\,\lambda\,|\psi\rangle &= \overline{\lambda} && \text{using eigenvector equation} \\
\lambda &= \overline{\lambda}
\end{aligned}
$$

With this we can conclude λ is real because if $a + \mathrm{i}b = a - \mathrm{i}b$ then $b = 0$ and hence the value is real.

Exercise 3

Show that $\hat{E}_N$ is a projector.

Solution. First, we show $\hat{E}_N$ is idempotent.

$$\begin{aligned}\hat{E}_N^2 &= \left(\sum_n^N |f_n\rangle\langle f_n|\right)\left(\sum_m^N |f_m\rangle\langle f_m|\right) \\ &= \sum_{n,m}^N (|f_n\rangle\langle f_n|)(|f_m\rangle\langle f_m|) \\ &= \sum_{n,m}^N |f_n\rangle \langle f_n|f_m\rangle \langle f_m| \\ &= \sum_{n,m}^N |f_n\rangle \delta_{n,m} \langle f_m| \\ &= \sum_n^N |f_n\rangle\langle f_n| = \hat{E}_N\end{aligned}$$

Now we need to show $\hat{E}_N$ is self-adjoint.

$$\begin{aligned}\hat{E}_N^\dagger &= \left(\sum_n^N |f_n\rangle\langle f_n|\right)^\dagger \\ &= \sum_n^N (|f_n\rangle\langle f_n|)^\dagger \\ &= \sum_n^N |f_n\rangle\langle f_n| && \text{by } (ab)^\dagger = b^\dagger a^\dagger \text{ and } |f\rangle^\dagger = \langle f|\end{aligned}$$

Exercise 4

Derive the expression for the abstract equation $|\varphi\rangle = \hat{g}\,|\psi\rangle$ in this basis.

Solution. This question is pretty confusing, so let's just try and doing exactly what is done in the lecture notes.

$$\begin{aligned}
\langle f|\phi\rangle &= \langle f|\,\hat{g}\mathbb{1}\,|\psi\rangle \\
&= \langle f|\,\hat{g}\left(\sum_n |f_n\rangle\langle f_n| + \int_J |f'\rangle\langle f'|\,\mathrm{d}f'\right)|\psi\rangle \\
&= \sum_n \langle f|\,\hat{g}\,|f_n\rangle\,\langle f_n|\psi\rangle + \int_J \langle f|\,\hat{g}\,|f'\rangle\,\langle f'|\psi\rangle\,\mathrm{d}f' \\
&= \sum_n g_m, n\psi_n + \int_J g(f,f')\psi(f)\,\mathrm{d}f' \\
&= \phi_m + \phi(f)
\end{aligned}$$

Not really sure what I'm doing here though, mostly symbol pushing.

Exercise 5

Look up and state the complete spectrum of the Hamiltonian of the Hydrogen atom (without spin or higher order corrections of any form). Give the exact source in a textbook.

Solution. The spectrum as given in [**shankar**] is given by the following.

$$E_n = -\frac{me^4}{2\hbar^2 n^2} \qquad n \in \mathbb{N}_{>0}$$

Exercise 6

Show that $\hat{U}$ is unitary.

Solution. First let $\hat{A} = \hat{f} - \mathrm{i}\mathbb{1}$. We can then write $\hat{U} = \hat{A}(\hat{A}^\dagger)^{-1}$. First thing we will show is that $\hat{A}$ is normal.

$$\hat{A}\hat{A}^\dagger = \left(\hat{f} - \mathrm{i}\mathbb{1}\right)\left(\hat{f} + \mathrm{i}\mathbb{1}\right) = \hat{f}^2 + \mathrm{i}\hat{f} - \mathrm{i}\hat{f} + \mathbb{1} = \hat{f}^2 + \mathbb{1}$$
$$\hat{A}^\dagger\hat{A} = \left(\hat{f} + \mathrm{i}\mathbb{1}\right)\left(\hat{f} - \mathrm{i}\mathbb{1}\right) = \hat{f}^2 - \mathrm{i}\hat{f} + \mathrm{i}\hat{f} + \mathbb{1} = \hat{f}^2 + \mathbb{1}$$

Now to show that $\hat{U}$ is unitary we need to show $\hat{U}\hat{U}^\dagger = \hat{U}^\dagger\hat{U} = \mathbb{1}$ so lets expand that out.

$$\begin{aligned} \hat{U}\hat{U}^\dagger &= \hat{A}\left(\hat{A}^\dagger\right)^{-1}\hat{A}^{-1}\hat{A}^\dagger \\ &= \hat{A}\left(\hat{A}\hat{A}^\dagger\right)^{-1}\hat{A}^\dagger \\ &= \hat{A}\left(\boxed{\hat{A}^\dagger\hat{A}}\right)^{-1}\hat{A}^\dagger \\ &= \hat{A}\hat{A}^{-1}\left(\hat{A}^\dagger\right)^{-1}\hat{A}^\dagger \\ &= \mathbb{1} \end{aligned}$$

$$\begin{aligned} \hat{U}^\dagger\hat{U} &= \hat{A}^{-1}\hat{A}^\dagger\hat{A}\left(\hat{A}^\dagger\right)^{-1} \\ &= \hat{A}^{-1}\left(\hat{A}^\dagger\hat{A}\right)^\dagger\left(\hat{A}^\dagger\right)^{-1} \\ &= \hat{A}^{-1}\left(\boxed{\hat{A}\hat{A}^\dagger}\right)^\dagger\left(\hat{A}^\dagger\right)^{-1} \\ &= \hat{A}^{-1}\hat{A}A^\dagger\left(\hat{A}^\dagger\right)^{-1} \\ &= \mathbb{1} \end{aligned}$$

Where in the colored box we've used the fact that $\hat{A}$ is normal. Since we've shown $\hat{U}\hat{U}^\dagger = \hat{U}^\dagger\hat{U} = \mathbb{1}$ we can conclude $\hat{U}$ is unitary.

Exercise 7

Show that each α is a complex number on the unit circle of the complex plane, i.e., that its modulus squared is $\alpha\alpha^* = 1$.

Solution. Let $|\psi\rangle$ be an eigenvector of $\hat{U}$.

$$\langle\psi|\psi\rangle = \langle\psi|\hat{U}^\dagger\hat{U}|\psi\rangle = \langle\psi|\alpha^*\alpha|\psi\rangle = |\alpha|^2 \langle\psi|\psi\rangle$$

With this, we conclude $\alpha = \mathrm{e}^{\mathrm{i}\theta}$ for some θ and hence α is on the complex unit circle.

Exercise 8

ssume that $\hat{f}$ is a self-adjoint operator with a purely point spectrum $\{f_n\}$. Further assume that $\hat{U}$ is the Cayley transform of f. Determine whether or not the operator $\hat{Q} := \hat{f} + \hat{U}$ is a normal operator. If not, why not? If yes, what is the spectrum of $\hat{Q}$?

Solution. For $\hat{Q}$ to be normal we must have $\hat{Q}\hat{Q}^\dagger = \hat{Q}^\dagger\hat{Q}$, so let's expand both sides to see what we get.

$$\begin{aligned} \hat{Q}\hat{Q}^\dagger &= \left(\hat{f} + \hat{U}\right)\left(\hat{f} + \hat{U}^\dagger\right) \\ &= \hat{f}^2 + \hat{f}\hat{U}^\dagger + \hat{U}\hat{f} + \mathbb{1} \end{aligned} \qquad \begin{aligned} \hat{Q}^\dagger\hat{Q} &= \left(\hat{f} + \hat{U}^\dagger\right)\left(\hat{f} + \hat{U}\right) \\ &= \hat{f}^2 + \hat{f}\hat{U} + \hat{U}^\dagger\hat{f} + \mathbb{1} \end{aligned}$$

So we can see if $\hat{f}\hat{U}^\dagger + \hat{U}\hat{f} = \hat{f}\hat{U} + \hat{U}^\dagger\hat{f}$ then $\hat{Q}$ is normal. Now because we can simultaneously diagonalize $\hat{f}$ and $\hat{U}$, we can make the followin expansions.

$$\begin{aligned} \hat{f}\hat{U}^\dagger + \hat{U}\hat{f} &= \left(\sum_n f_n \left|f_n\right\rangle\!\left\langle f_n\right|\right)\left(\sum_n u_n^* \left|f_n\right\rangle\!\left\langle f_n\right|\right) + \left(\sum_n u_n \left|f_n\right\rangle\!\left\langle f_n\right|\right)\left(\sum_n f_n \left|f_n\right\rangle\!\left\langle f_n\right|\right) \\ &= \sum_n f_n(u_n + u_n^*) \left|f_n\right\rangle\!\left\langle f_n\right| \\ &= \sum_n f_n u_n \left|f_n\right\rangle\!\left\langle f_n\right| + \sum_n u_n^* f_n \left|f_n\right\rangle\!\left\langle f_n\right| \\ &= \hat{f}\hat{U} + \hat{U}^\dagger\hat{f} \end{aligned}$$

Hence $\hat{Q}$ is normal. The spectrum of $\hat{Q}$ would be given by the intersection of the spectrum of $\hat{f}$ and $\hat{U}$ ($\mathrm{spec}(\hat{Q}) = \mathrm{spec}(\hat{f}) \cap \mathrm{spec}(\hat{U})$).

Exercise 1

Consider the harmonic oscillator of above, choose $L = (2m\omega)^{-1/2}$ and assume that the oscillator system is in this state:

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|E_0\rangle - |E_1\rangle)$$

Calculate $\overline{H}$ and $\overline{x}$ and the position wave function $\psi(x) := \langle x|\psi\rangle$.

Solution. Let's start with the expectation value of position.

$$\begin{aligned}
\overline{x} &= \sum_{n,m} \psi_n^* x_{n,m} \psi_m \\
&= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 0 & \cdots \end{bmatrix} L \begin{bmatrix} 0 & \sqrt{1} & & \\ \sqrt{1} & 0 & \sqrt{2} & \\ & \sqrt{2} & 0 & \\ & & & \ddots \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \\ \vdots \end{bmatrix} \\
&= \frac{L}{2} \begin{bmatrix} 1 & 1 & 0 & \cdots \end{bmatrix} \begin{bmatrix} \sqrt{1} \\ \sqrt{1} \\ \vdots \end{bmatrix} = L
\end{aligned}$$

Now the expectation value of the Hamiltonian.

$$\begin{aligned}
\overline{H} &= \sum_{n,m} \psi_n^* H_{n,m} \psi_m \\
&= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 0 & \cdots \end{bmatrix} \hbar\omega \begin{bmatrix} 0 & & & \\ & 1 + \frac{1}{2} & & \\ & & 2 + \frac{1}{2} & \\ & & & \ddots \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \\ \vdots \end{bmatrix} \\
&= \frac{\hbar\omega}{2} \begin{bmatrix} 1 & 1 & 0 & \cdots \end{bmatrix} \begin{bmatrix} 0 \\ 1 + \frac{1}{2} \\ \vdots \end{bmatrix} = \frac{3\hbar\omega}{4}
\end{aligned}$$

Lastly the wave function in the position basis.

$$\begin{aligned}
\psi(x) &= \langle x|\psi\rangle \\
&= \left(\sum_n \langle x|E_n\rangle \langle E_n|\right)\left(\frac{1}{\sqrt{2}}(|E_0\rangle - |E_1\rangle)\right) \\
&= \frac{1}{\sqrt{2}}(\langle E_0|x\rangle |E_0\rangle - \langle E_1|x\rangle |E_1\rangle) \\
&= \frac{\exp\left(\frac{-x^2}{4L^2}\right)}{\sqrt{2}\pi^{1/4}}\left(|E_0\rangle + \frac{x}{L} |E_1\rangle\right)
\end{aligned}$$