Advanced Quantum Theory Homework 5

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Due: Mon, Nov 2, 2020 11:50 AM
Course: AMATH 673

Exercise 2

Verify the canonical commutation relation in the position representation, i.e., verify that, for all (differentiable) wave functions $\psi(x)$:

$$(\hat{x}\hat{p} - \hat{p}\hat{x} - \mathrm{i}\hbar).\psi(x) = 0$$

Solution. Let's first evaluate each term separately.

$$(\hat{x}\hat{p}).\psi(x) = \hat{x}.(\hat{p}.\psi(x)) = \hat{x}.\left(-\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}x}\psi(x)\right) = -\mathrm{i}\hbar x\psi'(x)$$

$$(\hat{p}\hat{x}).\psi(x) = \hat{p}.(\hat{x}.\psi(x)) = \hat{p}.(x\psi(x)) = -\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}x}(x\psi(x)) = -\mathrm{i}\hbar\big(\psi(x) + x\psi'(x)\big)$$

$$(\mathrm{i}\hbar).\psi(x) = \mathrm{i}\hbar\psi(x)$$

Putting these together (with the appropriate signs) yields

$$\begin{aligned} -\mathrm{i}\hbar x\psi'(x) &+ \mathrm{i}\hbar\big(\psi(x) + x\psi'(x)\big) - \mathrm{i}\hbar\psi(x) \\ &= -\mathrm{i}\hbar x\psi'(x) + \mathrm{i}\hbar\psi(x) + \mathrm{i}\hbar x\psi'(x) - \mathrm{i}\hbar\psi(x) \\ &= 0 \end{aligned}$$

as desired. I hope you like my colors.

Exercise 3

Derive the action of $\hat{x}$ and $\hat{p}$ on momentum wave functions, i.e., derive the short hand notation $\hat{x}.\tilde{\psi}(p) = ?\tilde{\psi}(p)$ and $\hat{p}.\tilde{\psi}(p) = ?\tilde{\psi}(p)$ analogously to how we derived the short hand notation for the position representation.

Solution. Lets start with the simple one $\hat{p}.\tilde{\psi}(p)$. Begin by letting $|\phi\rangle = \hat{p}\,|\psi\rangle$.

$$\tilde{\phi}(p) = \langle p|\phi\rangle = \langle p|\,\hat{p}\,|\psi\rangle = p\,\langle p|\psi\rangle = p\tilde{\psi}(p)$$

With this we can conclude $\hat{p}.\tilde{\psi}(p) = p\tilde{\psi}(p)$.

Now to the slightly more challenging $\hat{x}.\tilde{\psi}(p)$. We'll first prove a little lemma we'll use in our computation down the line.

Lemma.

$$\langle p|\,\hat{x}\,|p'\rangle = \mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}p}\delta(p'-p)$$

Proof. Let's do two insertions of the identity in the position basis which we know how to handle a little bit better.

$$\begin{aligned}
\langle p|\,\hat{x}\,|p'\rangle &= \langle p|\,\mathbb{1}\hat{x}\mathbb{1}\,|p'\rangle \\
&= \int_{\mathbb{R}^2}\mathrm{d}x\,\mathrm{d}x'\,\langle p|x\rangle\,\langle x|\,\hat{x}\,|x'\rangle\,\langle x'|p'\rangle \\
&= \int_{\mathbb{R}^2}\mathrm{d}x\,\mathrm{d}x'\,\frac{\mathrm{e}^{-\mathrm{i}px/\hbar}}{\sqrt{2\pi\hbar}}\left[x\delta(x-x')\right]\frac{\mathrm{e}^{\mathrm{i}p'x'/\hbar}}{\sqrt{2\pi\hbar}} \\
&= \frac{1}{2\pi\hbar}\int_{\mathbb{R}}\mathrm{d}x\,x\mathrm{e}^{\mathrm{i}x(p'-p)/\hbar} \\
&= \frac{\mathrm{i}}{2\pi}\int_{\mathbb{R}}\mathrm{d}x\,\frac{\mathrm{d}}{\mathrm{d}p}\mathrm{e}^{\mathrm{i}x(p'-p)/\hbar} \\
&= \frac{\mathrm{i}}{2\pi}\frac{\mathrm{d}}{\mathrm{d}p}\int_{\mathbb{R}}\mathrm{d}x\,\mathrm{e}^{\mathrm{i}x(p'-p)/\hbar} \\
&= \frac{\mathrm{i}}{2\pi}\frac{\mathrm{d}}{\mathrm{d}p}\left[-2\pi\hbar\delta(p'-p)\right] \\
&= \mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}p}\delta(p-p')
\end{aligned}$$

Great, so now let's get back to the question at hand. Let $|\phi\rangle = \hat{x}\,|\psi\rangle$. Then we have

$$\begin{aligned}
\tilde{\phi}(p) &= \langle p|\phi\rangle = \langle p|\,\hat{x}\,|\psi\rangle \\
&= \int_{\mathbb{R}}\mathrm{d}p'\,\langle p|\,\hat{x}\,|p'\rangle\,\langle p'|\psi\rangle \\
&= \int_{\mathbb{R}}\mathrm{d}p'\,\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}p}\delta(p-p')\tilde{\psi}(p') = \mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}p}\tilde{\psi}(p)
\end{aligned}$$

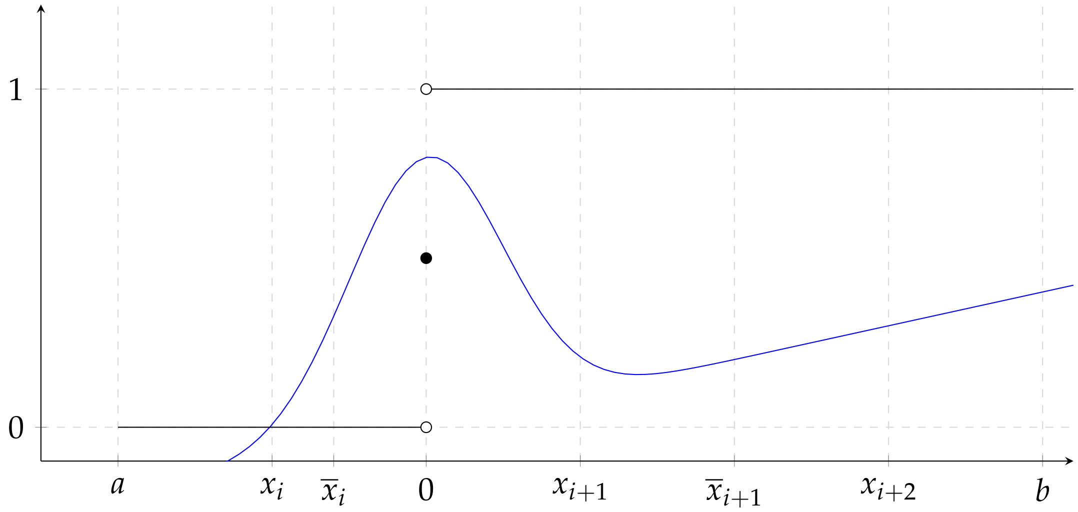
With this we can conclude $\hat{x}.\tilde{\psi}(p) = \mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}p}\tilde{\psi}(p)$, or perhaps if we're being technical $\hat{x}.\tilde{\psi}(p) = \mathrm{i}\hbar\frac{\partial}{\partial p}\tilde{\psi}(p)$. That has a nice symmetry[1] with $\hat{p}$ acting in position space.

[1] Presumably coming from the fact that x and p are Fourier transforms of each other?

Exercise 1

Explain Eq. 6.5 using a sketch of the plot of a function and a partitioning of the integration interval.

Solution. Using the following plot, we can calculate the Riemann-Stieltjes integral of the function in blue with respect the Heaviside function $\theta(x)$. Because $\theta(x)$ is the same



everywhere except near zero, all the terms in our summation cancel out and we are left with the following equation.

$$\int_a^b f(x)\mathrm{d}\theta(x) = \lim_{\varepsilon\to 0} f(\overline{x}_i)[\theta(x_{i+1}) - \theta(x_i)] = \lim_{\varepsilon\to 0} f(\overline{x}_i)$$

As ε goes to 0, $\overline{x}_i$ is forced to approach zero and hence in the limit $\int_a^b f(x)\mathrm{d}\theta(x) = f(0)$.

It's worth noting that if one of the interval points lands right on the origin $x_j = 0$, then the picture changes slightly because we have two terms.

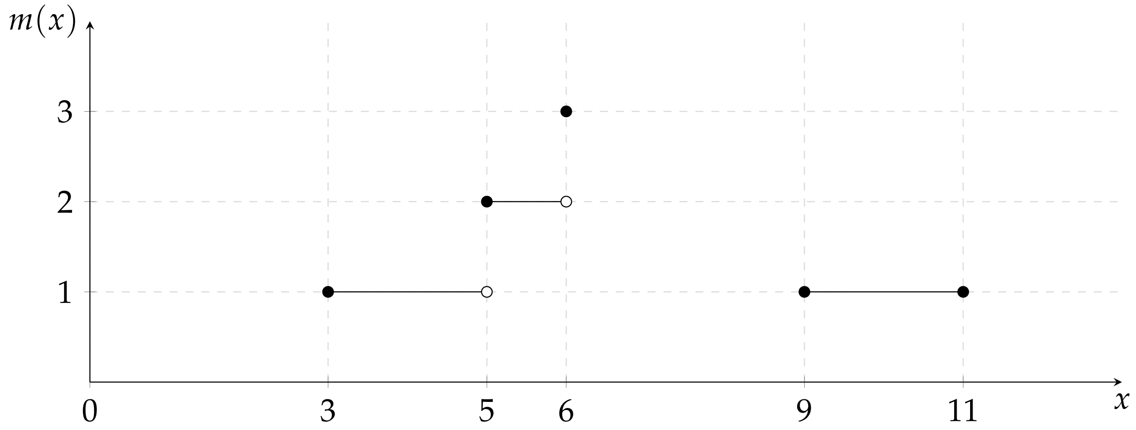
$$\int_a^b f(x)\mathrm{d}\theta(x) = \lim_{\varepsilon\to 0} f(\overline{x}_i)[\theta(x_{i+1}) - \theta(x_i)] = \lim_{\varepsilon\to 0} \frac{1}{2} f(\overline{x}_{j+1}) + \frac{1}{2} f(\overline{x}_j)$$

In the limit of small ε, this approaches the average of the points just left and right of 0, which is of course (for continuous functions) $f(0)$.

Exercise 2

Plot an integrator function $m(x)$ which integrates over the intervals $[3, 6]$ and $[9, 11]$ and sums over the values of the integrand at the points $x = 5$ and $x = 6$.

Solution. The following function $m(x)$ will pluck out the values at 5 and 6 so we can add them. I've only defined the function on $[3, 6] \cup [9, 11]$ because that's seems like the easiest thing to do.



Exercise 1

There are indications from studies of quantum gravity, that the uncertainty relation between positions and momenta acquire corrections due to gravity effects and should be of the form: $\Delta x \Delta p \geq \frac{\hbar}{2}(1+\beta(\Delta p)^2+\ldots)$, where β is expected to be a small positive number. Show that this type of uncertainty relation arises if the canonical commutation relation is modified to read$[\hat{x},\hat{p}] = \mathrm{i}\hbar(1+\beta\hat{p}^2)$. Sketch the modified uncertainty relation $\Delta x \Delta p \geq \frac{\hbar}{2}(1+\beta(\Delta p)^2)$ in the Δp versus Δx plane. Bonus: Show that this resulting uncertainty relation implies that the uncertainty in position can never be smaller than $\Delta x_{\text{min}} = \hbar\sqrt{\beta}$.

Solution. Let's start with the general uncertainty principle.

$$\Delta f \Delta g \geq \frac{1}{2}|\langle\psi|[f,g]|\psi\rangle|$$

We can now replace f and g with x and p respectively to obtain the new uncertainty principle.

$$\begin{aligned}
\Delta x \Delta p &\geq \frac{1}{2}\Big|\langle\psi|\mathrm{i}\hbar(1+\beta p^2)|\psi\rangle\Big| \\
&= \frac{\hbar}{2}\Big|\langle\psi|\psi\rangle + \beta\langle\psi|p^2|\psi\rangle\Big| \\
&= \frac{\hbar}{2}\Big(1+\beta\,\overline{p^2}\Big) \\
&= \frac{\hbar}{2}\Big(1+\beta(\Delta p)^2+\beta\,\overline{p}^2\Big) && \text{(using the definition of } \Delta p)
\end{aligned}$$

Or, written slightly differently we have $\Delta x \Delta p \geq \frac{\hbar}{2}(1+\beta(\Delta p)^2+\ldots)$.

Now if write this as $xy = \frac{\hbar}{2}(1+\beta y^2)$, it's maybe slightly easy to see it's a hyperbola. To find the minimum value for $x = \Delta x$, let's first solve for y.

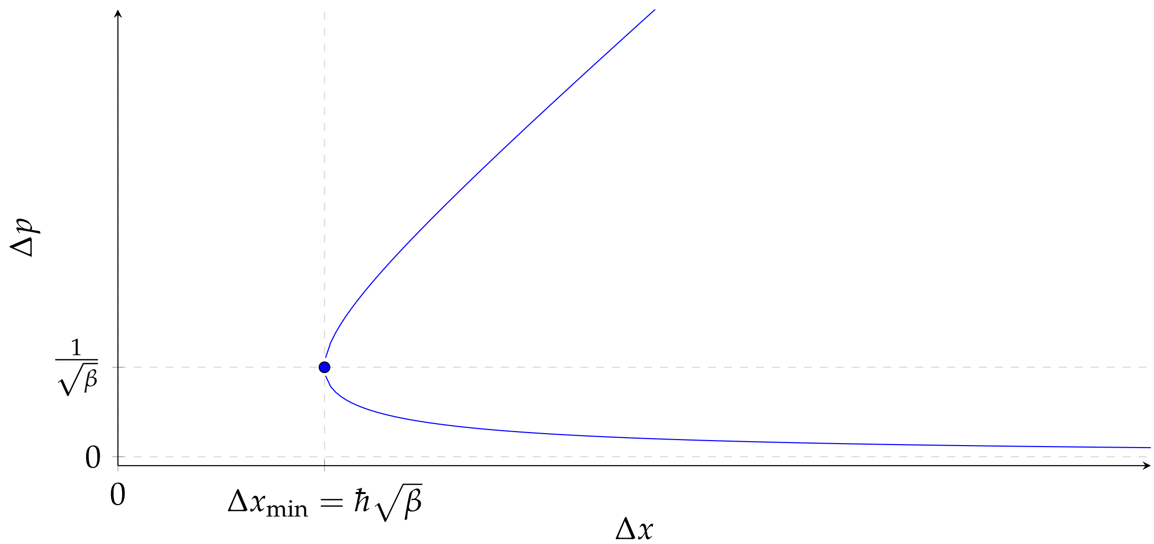
$$y^2 - \frac{2}{\hbar\beta}xy + \frac{1}{\beta} = 0 \implies y = \frac{\frac{2}{\hbar\beta}x \pm \sqrt{\frac{4}{\hbar^2\beta^2}x^2 - \frac{4}{\beta}}}{2} = \frac{x}{\hbar\beta} \pm \sqrt{\frac{x^2}{\hbar^2\beta^2} - \frac{1}{\beta}}$$

From here we can take the derivative of y and find where if evaluates to infinity. This is the point we're looking for.

$$y'\Big|_{y'\to\infty} = \frac{1}{\hbar\beta} + \frac{\frac{2x}{\hbar^2\beta^2}}{\sqrt{\frac{x^2}{\hbar^2\beta^2} - \frac{1}{\beta}}}\Bigg|_{y'\to\infty} \implies \frac{x^2}{\hbar^2\beta^2} = \frac{1}{\beta}$$

Where we've arrived at the condition $\Delta x_{\text{min}} = \hbar\sqrt{\beta}$. Putting the two together we have $\Delta x_{\text{min}}\Delta p = \hbar\sqrt{\beta}\frac{1}{\sqrt{\beta}} = \hbar$ which indeed satisfies the uncertainty principle. Nice.

Okay, to get onto the plotting. I couldn't figure out how to shade the region "inside" (to the right of the blue line), but that's the allowed region. Here we plot the portion of the hyperbola where both Δx and Δp are positive because those are the only physical values.


Δp
$\frac{1}{\sqrt{\beta}}$
0
0
$\Delta x_{\text{min}} = \hbar\sqrt{\beta}$
Δx

Exercise 2

Ultimately, every clock is a quantum system, with the clock's pointer or display consisting of one or more observables of the system. Even small quantum systems such as a nucleus, an electron, atom or molecule have been made to serve as clocks. Assume now that you want to use a small system, such as a molecule, as a clock by observing how one of its observables changes over time. Assume that your quantum clock possess a discrete and bounded energy spectrum $E_1 \leq E_2 \leq E_3 \leq \ldots \leq E_{\max}$ with $E_{\max} - E_1 = 1\text{eV}$ (1eV=1 electronvolt) which is a typical energy scale in atomic physics.

(a) Calculate the maximum uncertainty in energy, ΔE that your quantum clock can possess.

(b) Calculate the maximally achievable accuracy for such a clock. I.e., what is the shortest time interval (in units of seconds) within which any observable property of the clock could change its expectation value by a standard deviation?

Solution. **??** The maximum energy uncertainty would be the highest energy minus the lowest energy. I've gotta be missing something for this question... $\Delta E = E_{\max} - E_1 = 1\text{eV}$.

??

$$\Delta t \geq \frac{1}{\Delta H}\frac{\hbar}{2} = \frac{1}{1\text{eV}}\frac{\hbar}{2} = \frac{6.58 \times 10^{-16}\text{eV} \cdot \text{s}}{2\text{eV}} = 3.29 \times 10^{-16}\,\text{s}$$